A GLOBAL STEERING METHOD FOR NONHOLONOMIC SYSTEMS*

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EXTENDED ABSTRACT. Nonholonomic systems have attracted the attention of the scientific community for several years, due to the theoretical challenges they offer and the numerous important applications they cover. From the point of view of control theory, a nonholonomic system is a driftless control-affine system and is written as

$$(\Sigma) \quad \dot{x} = \sum_{i=1}^{m} u_i X_i(x), \ x \in \Omega, \quad u = (u_1, \dots, u_m) \in \mathbb{R}^m,$$

where Ω is an open connected subset of $\mathbb{R}^n$, and $X_1, \dots, X_m$ are C^∞ vector fields on Ω. Admissible inputs are $\mathbb{R}^m$-valued measurable functions $u(\cdot)$ defined on some interval $[0, T]$ and a trajectory of (Σ), corresponding to some $x_0 \in \Omega$ and to an admissible input $u(\cdot)$, is the (maximal) solution $x(\cdot)$ in Ω of the Cauchy problem defined by $\dot{x}(t) = \sum_{i=1}^{m} u_i(t) X_i(x(t))$, $t \in [0, T]$, and $x(0) = x_0$.

In this paper, we address the *motion planning problem* (MPP for short) for (Σ), namely determine a procedure which associates with every pair of points $(p, q) \in \Omega \times \Omega$ an admissible input $u(\cdot)$ defined on some interval $[0, T]$, such that the corresponding trajectory of (Σ) starting from p at $t = 0$ reaches q at $t = T$. As for the existence of a solution to MPP, this is equivalent to the complete controllability of (Σ). After the works of Chow and Rashevsky in the thirties, and that of Sussmann and Stefan in the seventies, the issue of complete controllability for nonholomic systems is well-understood and it is usually guaranteed by assuming that the Lie Algebraic Rank Condition (also known as the Hörmander condition) holds for (Σ). This easily checkable condition is not only sufficient for complete controllability but also necessary when the vector fields are analytic. From a practical viewpoint, assuming the LARC is, in a sense, the minimal requirement to ensure complete controllability for (Σ) and this is what we will do for all the control systems considered hereafter.

As for the construction of the solutions of the MPP, we present, in this paper, a complete procedure solving the MPP for a nonholonomic system subject to the sole LARC. By "complete procedure", we mean that the following properties must be guaranteed by the proposed procedure.

1. Global character of the algorithm: for every pair of points (p, q) in Ω, the algorithm must produce a steering control. (Note that the core of many

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algorithms consists in a local procedure and turning the latter into a global one is not always a trivial issue.)

2. Proof of convergence of the algorithm.
3. Regarding numerical implementations, no prohibitive limitation on the state dimension n.
4. Usefulness for practical applications, e.g., robustness with respect to the dynamics, “nice” trajectories produced by the algorithm, (no cusps neither large oscillations), and possibility of localizing the algorithm in order to handle obstacles (i.e., reducing the working space Ω to any smaller open and connected subset of $\mathbb{R}^n$).

We now describe in a condensed manner the global motion planning strategy developed in this paper. The latter is presented as an algorithmic procedure associated with a given nonholonomic system (Σ) defined on $\Omega \subset \mathbb{R}^n$. The required inputs are initial and final points x^{initial} and x^{final} belonging to Ω, a tolerance $e > 0$, and a compact convex set $K \subset \Omega$ (of appropriate size) equal to the closure of its interior which is a neighborhood of both x^{initial} and x^{final}. For instance, K can be chosen to be a large enough compact tubular neighborhood constructed around a curve joining x^{initial} and x^{final}. The global steering method is summarized in Algorithm 1.

Algorithm 1 Global Approximate Steering Algorithm: Global$(x^{\text{initial}}, x^{\text{final}}, e, K)$

Build a decomposition of K into a finite number of compacts sets $\mathcal{V}^c_{\mathcal{J}_i}$, with $i = 1, \ldots, M$.
Construct the connectedness graph $\mathsf{G} := (\mathsf{N},\ \mathsf{E})$ associated with this decomposition and choose a simple path $\mathsf{p} := \{j_0, j_1 \ldots, j_{\bar{M}}\}$ in G such that $x^{\text{initial}} \in \mathcal{V}^c_{\mathcal{J}_{j_0}}$ and $x^{\text{final}} \in \mathcal{V}^c_{\mathcal{J}_{j_{\bar{M}}}}$.
Choose a sequence $(x^i)_{i=1,\ldots,\bar{M}-1}$ such that $x^i \in \mathcal{V}^c_{\mathcal{J}_{j_i}} \cap \mathcal{V}^c_{\mathcal{J}_{j_{i+1}}}$.
Set $x := x^{\text{initial}}$.
for $i = 1, \ldots, \bar{M} - 1$ **do**
    Apply the Desingularization Algorithm at $a := x^i$ with $\mathcal{J} := \mathcal{J}_i$.
    {the output is an m-tuple of vector fields ξ on $\mathcal{V}_{\mathcal{J}_i} \times \mathbb{R}^{\widetilde{n}}$ which is free up to step r.}
    Let AppSteer be the LAS method associated to the approximation $\mathcal{A}^\xi$ of ξ on $\mathcal{V}_{\mathcal{J}_i} \times \mathbb{R}^{\widetilde{n}}$ and to its steering law $\text{Exact}_{m,r}$.
    Set $\widetilde{x}_0 := (x, 0)$, $\widetilde{x}_1 := (x^i, 0)$, and $\mathcal{V}^c := \mathcal{V}^c_{\mathcal{J}_i} \times \overline{B}_R(0)$ with $R > 0$ large enough.
    Apply GlobalFree$(\widetilde{x}_0, \widetilde{x}_1, e, \mathcal{V}^c, \text{AppSteer})$ to ξ.
    {the algorithm stops at a point $\widetilde{x}$ which is e-close to $\widetilde{x}_1$;}
    return $x := \pi(\widetilde{x})$.
    {$\pi : \mathcal{V}_{\mathcal{J}_i} \times \mathbb{R}^{\widetilde{n}} \to \mathcal{V}_{\mathcal{J}_i}$ is the canonical projection.}
end for

The paper is devoted to the construction of the various steps of this algorithm. We will also show that each of these steps is conceived so that the overall construction is a complete procedure in the sense defined previously. In particular, the convergence issue is addressed in the following theorem.

Theorem 0.1. *Let (Σ) be a nonholonomic system on $\Omega \subset \mathbb{R}^n$ satisfying the LARC. For every $e > 0$, every connected compact set K which is equal to the closure of its interior, and every pair of points $(x^{\text{initial}}, x^{\text{final}})$ in the interior of K, Algorithm 1* *steers, in a finite number of steps, the control system* (Σ) *from* x^{initial} *to a point* $x \in K$ *such that* $d(x, x^{\text{final}}) < e$.

Finally, we mention possible extensions of our algorithm. The first one concerns the working space Ω. Since it is an arbitrary open connected set of $\mathbb{R}^n$, one can extend the algorithm to the case where the working space is a smooth connected manifold of finite dimension. From a numerical point of view, there would be the additional burden of computing the charts. A second extension deals with the stabilization issue. Indeed, at the heart of the algorithm lies an iterative procedure, which can be easily adapted for stabilization tasks. Another possible generalization takes advantage of devising from our algorithm a globally regular input, one can then address the motion planning of dynamical extensions of the nonholonomic control systems considered in the present paper. Finally, let us point out the modular nature of Algorithm 1: one can propose other approaches to obtain uniformly contractive local methods (other desingularization methods or different ways of dealing with singular points), or replace $\text{Exact}_{m,r}(\cdot)$ by more efficient control strategies for general nilpotent systems. The full paper and references are available at `http://uma.ensta.fr/publication.php?id=1004`.