RATIONAL REPRESENTATIONS AND MINIMAL STATE REPRESENTATIONS OF BEHAVIORS

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EXTENDED ABSTRACT. In this paper, we deal with a special category of representations of linear time invariant differential systems (LTIDS) in the behavioral framework. In the behavioral framework, a behavior, $\mathfrak{B}$, of a LTIDS admits many representations such as the kernel of a polynomial differential operator [2], the kernel of a rational differential operator [5], etc. In modeling of LTIDS, we often need variables other than the manifest ones, called latent variables. Representations involving these are called *latent variable representations*. These latent variable representations in general involve polynomial, and/or rational differential operators. There is also a special class of latent variable representations called *state representations*, in which the latent variable satisfies a special property called the axiom of state. This axiom states that this variable has the property that it parametrizes the memory of the system, i.e., it splits the past and future of the behavior. A behavior admits different kinds of state representations such as classical *input/state/output (I/S/O) representations*, *driving variable representations*, *output nulling representations* [4], [1]. In this paper we shall limit our attention to driving variable and output nulling representations.

State representations and rational representations. In the behavioral framework, a linear time-invariant differential system is defined as a triple $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$, where $\mathbb{R}$ is the time-axis, $\mathbb{R}^{\mathtt{w}}$ is the signal space and $\mathfrak{B} \subset \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ is the behavior. We denote set of all linear differential systems with $\mathtt{w}$ variables by $\mathfrak{L}^{\mathtt{w}}$. As mentioned earlier, it is shown in [5] that a behavior $\mathfrak{B}$ of a LTIDS admits a rational kernel representation, i.e., $\mathfrak{B} = \ker(G(\frac{\mathrm{d}}{\mathrm{d}t}))$, where $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$. Controllable behaviors also admit a rational image representation i.e., $\mathfrak{B} = \mathrm{im}(H(\frac{\mathrm{d}}{\mathrm{d}t}))$, where $H \in \mathbb{R}(\xi)^{\mathtt{m} \times \bullet}$. It is also shown that G, H appearing in the above mentioned representations can be chosen of full row rank and full column rank, respectively. We call representations with such G, H matrices minimal rational kernel and minimal rational image representations, respectively. We denote the ring of proper real rational functions by $\mathbb{R}(\xi)_{\mathcal{P}}$ and $\mathbb{R}(\xi)_{\mathcal{P}}^{\bullet \times \bullet}$ denotes proper real rational matrices of appropriate dimensions.

It is shown in [4] that a behavior, $\mathfrak{B}$, admits a state representation,

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}x &= Ax + Bw, \\ 0 &= Cx + Dw, \end{aligned} \tag{0.1}$$

such that $\mathfrak{B} = \{w \mid \exists x \text{ such that (0.1) holds }\}$, where (A, B, C, D) are matrices of appropriate dimensions and $(w, x) \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \times \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$. We call (0.1) an output nulling representation, $\mathfrak{B}_{ON}(A, B, C, D)$, of $\mathfrak{B}$.

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It is also shown in [4] that a behavior, $\mathfrak{B}$, admits a state representation,

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}x &= Ax + Bv, \\ w &= Cx + Dv, \end{aligned} \tag{0.2}$$

such that $\mathfrak{B} = \{w \mid \exists (x, v) \text{ such that (0.2) holds }\}$, where (A, B, C, D) are matrices of appropriate dimensions and $(w, x, v) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{v}})$. We call (0.2) a driving variable representation, $\mathfrak{B}_{DV}(A, B, C, D)$, of $\mathfrak{B}$. We call v in (0.2) a driving variable.

At this juncture, it is very natural to pose a question regarding the relation between the above mentioned state representations and rational representations of a given behavior. The following theorem illustrates the relation between a proper real rational matrix appearing in a rational kernel representation and an output nulling representation yielded from its realization.

THEOREM 0.1. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, $\mathfrak{B} = \ker(G(\frac{\mathrm{d}}{\mathrm{d}t}))$, and G be a proper real rational matrix of full row rank. Let $G(\xi) = C(\xi I - A)^{-1}B + D$ be a realization of G, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{w}}, C \in \mathbb{R}^{\mathtt{n}_1\times\mathtt{n}}, D \in \mathbb{R}^{\mathtt{n}_1\times\mathtt{w}}$. Then $\mathfrak{B}_{ON}(A, B, C, D)_{\mathrm{ext}} = \ker(G(\frac{\mathrm{d}}{\mathrm{d}t}))$ if (A, B) is a controllable pair.* From the above theorem we have that controllability of the pair (A, B) is sufficient for $\mathfrak{B}_{ON}(A, B, C, D)_{\mathrm{ext}} = \ker(G(\frac{\mathrm{d}}{\mathrm{d}t}))$. In the following example we show that it is not a necessary condition.

EXAMPLE 0.2. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$, and $D = 1$. Clearly (A, B) is not a controllable pair. Direct computation yields $\mathfrak{B}_{ON}(A, B, C, D)_{\mathrm{ext}} = \ker(P(\frac{\mathrm{d}}{\mathrm{d}t}))$, where $P(\xi) = \xi$. We have $G(\xi) = C(\xi I - A)^{-1}B + D = \frac{\xi}{\xi - 1}$, so $\ker(G(\frac{\mathrm{d}}{\mathrm{d}t})) = \ker(P(\frac{\mathrm{d}}{\mathrm{d}t}))$.

Similarly, the following theorem illustrates the relation between a proper real rational matrix appearing in a rational image representation and a driving variable representation yielded from its realization.

THEOREM 0.3. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{contr}}$. Let $\mathfrak{B} = \mathrm{im}(H(\frac{\mathrm{d}}{\mathrm{d}t}))$ be a rational image representation, and H be a proper real rational matrix of full column rank. Let $H(\xi) = C(\xi I - A)^{-1}B + D$ be a realization of H, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}_2}, C \in \mathbb{R}^{\mathtt{w}\times\mathtt{n}}, D \in \mathbb{R}^{\mathtt{w}\times\mathtt{n}_2}$. Then, $\mathfrak{B}_{DV}(A, B, C, D)_{\mathrm{ext}} = \mathrm{im}(H(\frac{\mathrm{d}}{\mathrm{d}t}))$ if (A, B) is a controllable pair.*

As in the case of rational kernel representations, given a realization of $H(\xi) = C(\xi I - A)^{-1}B + D$, controllability of the pair (A, B) is not a necessary condition for $\mathfrak{B}_{DV}(A, B, C, D)_{\mathrm{ext}}$ to be equal to $\mathrm{im}(H(\frac{\mathrm{d}}{\mathrm{d}t}))$. This is illustrated by the following example.

EXAMPLE 0.4. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$, and $D = 1$. Then we have $\mathfrak{B}_{DV}(A, B, C, D)_{\mathrm{ext}} = \{w \mid \exists v \text{ such that } \frac{\mathrm{d}}{\mathrm{d}t}v = (\frac{\mathrm{d}}{\mathrm{d}t} - 1)w\} = \mathrm{im}\, H(\frac{\mathrm{d}}{\mathrm{d}t})$, where $H(\xi) = \frac{\xi}{\xi - 1}$.

Minimality of representations. Minimality of a representation is defined differently for each of the state space representations that a behavior admits [4]. An I/S/O representation is minimal if the state dimension is minimal over all I/S/O representations of $\mathfrak{B}$. A driving variable representation is minimal if the state and the driving variable dimensions are minimal over all driving variable representation of $\mathfrak{B}$. An output nulling representation is minimal if the state dimension and the dimension of the output space are minimal over all output nulling representations of $\mathfrak{B}$. It is well known that every proper real rational matrix admits a realization such that the underlying constant real matrices form a controllable and an observable pair. In the classical I/S/O setting, a representation obtained from such realization is always minimal. However this is not the case with driving variable and output nulling representations. The controllability and observability conditions on the underlying constant real matrices are not sufficient to ensure minimality of the output nulling and driving variable representations.

This motivates us to pose the following question: Given are a behavior $\mathfrak{B}$, its rational kernel representation with a proper real rational matrix and a realization of the above matrix such that the underlying constant real matrices form a controllable and an observable pair. Find conditions under which the above realization yields a minimal output nulling representation of the behavior. A similar question is relevant in the case of driving variable representations too. The following theorem gives conditions on $G \in \mathbb{R}(\xi)_{\mathcal{P}}^{\mathtt{n}_1 \times \mathtt{w}}$ so that its realization yields a minimal output nulling representation of $\mathfrak{B} = \ker(G(\frac{\mathrm{d}}{\mathrm{d}t}))$.

Theorem 0.5. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, $\mathfrak{B} = \ker(G(\frac{\mathrm{d}}{\mathrm{d}t}))$, and G be a proper real rational matrix of full row rank. Let $G(\xi) = C(\xi I - A)^{-1}B + D$ be a realization of G, where $A \in \mathbb{R}^{\mathtt{n} \times \mathtt{n}}, B \in \mathbb{R}^{\mathtt{n} \times \mathtt{w}}, C \in \mathbb{R}^{\mathtt{n}_1 \times \mathtt{n}}, D \in \mathbb{R}^{\mathtt{n}_1 \times \mathtt{w}}$, (A, B) controllable and (C, A) observable. Then $\mathfrak{B}_{ON}(A, B, C, D)$ is a minimal output nulling representation of $\mathfrak{B}$ if and only if $G(\xi)$ is left prime over $\mathbb{R}(\xi)_{\mathcal{P}}$.*

We shall now illustrate the previous theorem using the following example:

Example 0.6. Let $G(\xi) = \begin{bmatrix} \frac{\xi+1}{\xi-2} & 0 \end{bmatrix}$. We have $G(\xi) = C(\xi I - A)^{-1}B + D$, where $A = 2, B = \begin{bmatrix} \sqrt{3} & 0 \end{bmatrix}$, $C = \sqrt{3}$, $D = \begin{bmatrix} 1 & 0 \end{bmatrix}$. Clearly, G is left prime over $\mathbb{R}(\xi)_{\mathcal{P}}$. Therefore the realization yields a minimal output nulling representation of $\ker G(\frac{\mathrm{d}}{\mathrm{d}t})$.

The following theorem gives conditions on $H \in \mathbb{R}(\xi)_{\mathcal{P}}^{\mathtt{w} \times \mathtt{n}_2}$ so that its realization yields a minimal driving variable representation of $\mathfrak{B} = \mathrm{im}(H(\frac{\mathrm{d}}{\mathrm{d}t}))$.

Theorem 0.7. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{contr}}$. Let $\mathfrak{B} = \mathrm{im}(H(\frac{\mathrm{d}}{\mathrm{d}t}))$ be a rational image representation, and H be a proper real rational matrix of full column rank. Let $H(\xi) = C(\xi I - A)^{-1}B + D$ be a realization of H, where $A \in \mathbb{R}^{\mathtt{n} \times \mathtt{n}}, B \in \mathbb{R}^{\mathtt{n} \times \mathtt{n}_2}, C \in \mathbb{R}^{\mathtt{w} \times \mathtt{n}}, D \in \mathbb{R}^{\mathtt{w} \times \mathtt{n}_2}$, such that (A, B) controllable and (C, A) observable . Then, $\mathfrak{B}_{DV}(A, B, C, D)$ is a minimal driving variable representation of $\mathfrak{B}$ if and only if $H(\xi)$ has no zeros and is right prime over $\mathbb{R}(\xi)_{\mathcal{P}}$.*

We illustrate the previous theorem with the following example:

Example 0.8. Let $H(\xi) = \begin{bmatrix} \frac{\xi+1}{\xi-2} \\ 1 \end{bmatrix}$. We have $H(\xi) = C(\xi I - A)^{-1}B + D$, where $A = 2, B = \sqrt{3}, C = \begin{bmatrix} \sqrt{3} \\ 0 \end{bmatrix}, D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Clearly H is right prime over $\mathbb{R}(\xi)_{\mathcal{P}}$ and it has no zeros, and D has full column rank. It is easily verified that (A, B, C, D) is strongly observable. Hence using the above theorem, we conclude that the realization of $H(\xi)$ yielded a minimal driving variable representation of $\mathrm{im}(H(\frac{\mathrm{d}}{\mathrm{d}t}))$.

For a detailed discussion on the proofs and results, readers are referred to [3].

REFERENCES

[1] M. Kuiper, *First-Order Representaions of Linear Systems*, Birkhäuser, 1994.
[2] J.W. Polderman and J.C. Willems, *Introduction to Mathematical Systems Theory: a Behavioral Approach*, Springer-Verlag, Berlin, 1997.

[3] S.V. Gottimukkala, H.L. Trentelman and S. Fiaz " Realization and elimination in rational representations of behaviors, " manuscript submitted for publication, 2012.

[4] J.C. Willems, "Input-output and state-space representations of finite-dimensional linear time-invariant systems," *Linear Algebra and its Applications*, Vol. 50, pp. 581-608, 1983.

[5] J.C. Willems and Y. Yamamoto, "Behaviors defined by rational functions," *Linear algebra and its applications*, Vol. 425, pp. 226-241, 2007.