Controllability on $\mathrm{Sl}(2,\mathbb{C})$

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Abstract

The set $G = \mathrm{Sl}(2,\mathbb{C})$ of complex matrices of order two and determinant 1 has rank 1 and induces just one flag $S^2 = \mathbb{C}P^2$. In this case, the complex plane $\mathbb{C}$ is an open Bruhat cell given by the complex lines in $\mathbb{C}^2$ generated by vectors of type $(z, 1)$. The action of $g \in \mathrm{Sl}(2,\mathbb{C})$ on $z \in \mathbb{C}$ is given by the Möbius function.

The semigroup S of a transitive invariant control system $\Sigma : X + uY$, $|u| \leq 1$ on G is generated by the exponential of elements in the cone $\{\lambda(X + uY) : \lambda \geq 0,\ |u| \leq 1\}$. However, by convexity S is generated just by two half-lines which define the boundaries of the cone, i. e., $\exp t(X + Y)$ and $\exp t(X - Y)$, $t \geq 0$. Therefore, the controllability problem of Σ on G consist in to decide for which pairs of elements A, B in the Lie algebra $\mathbf{g} = \mathfrak{sl}(2,\mathbb{C})$ of G, the semigroup generated by $\exp tA$ and $\exp tB$, $t \geq 0$ is proper or not.

A semigroup of $\mathrm{Sl}(2,\mathbb{C})$ is proper if and only if its invariant control set in the flag $S^2 = \mathbb{C} \cup \{\infty\}$ is proper, [1]. Which means that there exists

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a proper invariant set by $\exp tA$ and $\exp tB$, $t \geq 0$. Furthermore, the pair A, B is controllable if the semigroup generated by $\exp tA$ and $\exp tB$, $t \geq 0$ is $\mathrm{Sl}(2, \mathbb{C})$.

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A matrix $A \in \mathrm{Sl}(2, \mathbb{C})$ defined by its rows (α, β) and $(\gamma, -\alpha)$ induces a vector field A on the sphere S^2 such that over $\mathbb{C}$ correspond to the quadratic vector field $A(z) = -\gamma z^2 + 2\alpha z + \beta$, associated to a Ricatti differential equation.

Any Möbius function is a composition of two types of functions: $z \mapsto az + b$ and $z \mapsto 1/z$ which preserves the set of circles in $\mathbb{C} \cup \{\infty\}$. Here the lines of $\mathbb{C}$ are the circles through ∞.

1 Trajectories of diagonalizable elements

Let B be a diagonalizable matrix given by

$$g \begin{pmatrix} -\gamma & 0 \\ 0 & \gamma \end{pmatrix} g^{-1}.$$

Proposition 1 *The trajectories of B are given by the images by g of the corresponding trajectories of $diag(-\gamma, \gamma)$. There are two fixed points $g(0)$ and $g(\infty)$ and*

1. If $\pm\gamma \in i\mathbb{R}$ the orbits are the circles $gC(0, r)$ where $C(0, r)$ is the circle with center 0 and radio r.
2. If $\pm\gamma \in \mathbb{R}$ the orbits remain in the circles through the fixed points $g(0)$ and $g(\infty)$. Inside of any circle there are four orbits: two fixed points and two connected component separates by the singularities. The trajectories moves from $g(\infty)$ to $g(0)$ if $\gamma < 0$, and in the contrary if $\gamma > 0$.

Any circle $gC(0, r)$ is called circle of level of B. The only one circle through ∞ is a line called *separator*

The Jordan canonical forms of a matrix in $\mathfrak{sl}(2, \mathbb{C})$ are $\begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Thus, there exists just one nilpotent orbit and the diagonalizable orbits are parametrized by the eigenvalues $a \in \mathbb{C}$. Just observe, that $A, B \in \mathfrak{sl}(2, \mathbb{C})$ and $g \in \mathrm{Sl}(2, \mathbb{C})$ the pair A, B is controllable if and only if the pair gAg^{-1}, gBg^{-1} is controllable. So, it is possible to select suitable pairs in a same adjoint orbit in $\mathfrak{sl}(2, \mathbb{C})^2$.

Proposition 2 *The standard form of a pair of diagonalizable matrices A, B is given by*

$$A = \begin{pmatrix} -\alpha & 0 \\ 0 & \alpha \end{pmatrix}, \; B = \begin{pmatrix} -\gamma - 2x\gamma & 2(\gamma + x\gamma) \\ -2x\gamma & \gamma + 2x\gamma \end{pmatrix}, \; Re\alpha, Re\gamma \geq 0.$$

Here, $\pm\gamma$ are the eigenvalues of B, $1 \in \mathbb{CP}^1$ is an eigenspace and $x = \frac{1}{w-1}$ where $w \in \mathbb{C} \subset \mathbb{CP}^1$ is the other eigenspace. Furthermore, 1 is the attractor if $Re(\gamma) > 1$.

2 Controllability

Since Σ is transitive $\mathcal{L}_{\mathbb{R}} = \mathfrak{sl}(2,\mathbb{C})$, thus there exists just one invariant control set in S^2, denoted by C with $\text{int}C \neq \emptyset$. Both, C and $\text{int}C$ are connected and Σ is controllable if and only if $C = S^2$.

Theorem 3 *Let Σ be a control system with A, B in the standard form. We have*

a) If A and B have real eigenvalue Σ is not controllable

b) If $\alpha \in i\mathbb{R}$ the only on invariant control set is the ring

$$C = \{z \in \mathbb{C} \cup \{\infty\} : \rho_1 \leq |z| \leq \rho_2\}, 0 \leq \rho_1 < \rho_2 \leq \infty$$

c) If the eigenvalues of A and B belong to $i\mathbb{R}$ the system Σ is controllable

d) If $\alpha \in i\mathbb{R}$, Σ is controllable if and only if $(exptB)K \subset K$, $t \geq 0$, where $K = B[0,\rho], \rho > 1$ if $|w| > 1$, *or* $K = B[0,\rho]^c, \rho < 1$ if $|w| < 1$. As a consequence, Σ is controllable if and only if B leave invariant a ball or the complement of a ball depending on $|w| > 1$ or $|w| < 1$.

In the sequel we establish conditions in order to a diagonalizable matrix X leave invariant a ball. This allows to us to give necessary and sufficient conditions to controllability. Assume the eigenvalues of X are $\pm\,\delta$ with $Re(\delta) > 0$, with w^+ (attractor) and w^- (repulsor) its fixed points. The following balls are $\exp(tX)$-invariant $t \geq 0$: half-planes limited by lines and closed balls with center ∞. Both, are images of compact balls for Moebius transformations.

Proposition 4 *The ball $B[0,r]$ is not invariant for X in the next two cases*

a) $w^- \in int(B[0,r])$

b) $w^+ \notin B[0,r]$.

Conclusion 5 *The previous results show a way to determine completely the controllability property of an invariant control systems on* $\mathrm{Sl}(2,\mathbb{C})$. *At the very end, it is necessary to find the subsets of* $\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$*, which are conformed by controllable or not controllable pairs. It is well known that the class $X + uY$ of systems is generically controllable with non restrict admissible controls. However, when restrict control are considered as $|u| \leq \rho$, it is necessary to find the values of ρ to decide when the pair $(X - \rho Y, X + \rho Y)$ is controllable or not. Specially, the bifurcation point ρ^* such that $(X - \rho Y, X + \rho Y)$ is not controllable for $\rho \leq \rho^*$ and controllable for $\rho > \rho^*$. This is possible with an explicitly description of the controllable pairs in* $\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$.

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