Limit behavior of control systems from shadowing semigroups and flows

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April 21, 2012

Abstract

The limit behavior of a class of control systems can be recovered from a semigroup actions on flag manifolds by the shadowing technique applied to semigroups and then to flows.

AMS 2000 Subject Classification: 37B35, 37B25, 93D30
Key words: Shadowing semigroups, control systems, Iwasawa decomposition, flag manifolds, Weyl group.

1 Introduction

Let $\Sigma = (M,\mathcal{U})$ be an affine control system on a Riemannian differentiable manifold M and an admissible class of control $\mathcal{U} \subset \mathbb{L}_\infty(\mathbb{R}, U \subset \mathbb{R}^m)$, given by

$$\dot{x}(t) = X\left(x(t)\right) + \sum_{j=1}^{m} u_j(t) Y^j\left(x(t)\right), \quad x(t) \in M.$$

*This research was partially supported by Proyectos FONDECYT n° 1100375
†Research partially supported by CNPq grant 30.10.60/94-0

U is compact, convex. Σ is locally accessible from the Lie algebra rank condition. Denote by $\pi(x,u,t)$ the solution of Σ with initial condition x, control $u \in \mathcal{U}$ and time t.

There exist a way to analyze the global dynamic behavior of Σ through the control flow Φ induced by the system on the fiber bundle $\mathcal{U} \times M$. Therefore, it is possible to apply the theory of dynamical systems to Φ to get relevant information for Σ.

2 Main Results

In [3] the authors study extensively this idea and prove

Theorem 1 *(1) The control sets of Σ correspond to the topologically mixing components of Φ. In particular, Σ is controllable if and only if Φ is topologically mixing. (2) The lift to $\mathcal{U} \times M$ of a chain control set $E \subset M$ of Σ, given by*

$$\mathcal{E} = \{(u,x) \in \mathcal{U} \times M : \varphi(t,x,u) \in E,\ t \in \mathbb{R}\},$$

is maximal invariant chain transitive set for Φ. Reciprocally, let $\mathcal{E}$ be a maximal invariant chain transitive set for Φ. Then, $\pi_M(\mathcal{E})$ is a chain control set. Here, $\pi_M : \mathcal{U} \times M \to M$ is the projection.

In this frame, a special attention is given on systems evolving on projective spaces because of the interest in the study of the linearized flow and thus in the study of the Lyapunov stability properties of a system. As a generalization, one is interested in looking at the control sets and chain control sets for the action of linear semigroups in projective spaces. Related to the control sets, this analysis was pursued in [1, 2, 5, 6] (see also [4] and [7]) where the authors study the action of semigroups in non compact semi-simple Lie groups on the flag manifolds of the group. The main technique is based on the fact that it is possible to characterize the chain control sets as intersections of control sets for semigroups $S_{\epsilon,A}$ generated by the ε-perturbation of A

$$(A,\epsilon) = \{g \in G : \exists h \in A, d'(h,g) < \epsilon\},$$

where d' is the uniform metric. Actually, the approach consists in shadowing the semigroup S.

Let $\mathbf{g}$ be the Lie algebra of G and $\mathbf{g} = \mathbf{k}+\mathbf{a}+\mathbf{n}^+$ the Iwasawa decomposition of $\mathbf{g}$. Contained in $\mathbf{a}$ there are the Weyl chambers. Associated with a chamber $\mathbf{a}^+$ there is a system of positive roots, denoted by Δ^+. The simple system of roots generating Δ^+ is denoted by Π. For a subset $\Theta \subset \Pi$ denote by P_Θ the parabolic subgroup defined by Θ. The associated flag manifold is $B_\Theta = G/P_\Theta$ and $B = G/P$ the maximal flag manifold. Let W be the Weyl group $W = M^*/M$, M^* is the normalizer and M the centralizer of $A = \exp \mathbf{a}$ in $K = \exp \mathbf{k}$.

Theorem 2 *For each $w \in W$ there exists a chain control set E_w for S on B.*

The set $W_{\mathcal{F}}(S) = \{w \in W : E_w = E_1\}$ is a subgroup of W and

$$W_{\mathcal{F}}(S)w_1 = W_{\mathcal{F}}(S)w_2 \Leftrightarrow E_{w_1} = E_{w_2}.$$

On the maximal flag manifold B the number of chain control sets equals the number of elements in the coset space $W_{\mathcal{F}}(S)\backslash W$ while the number of chain control sets on the flag B_Θ equals the order of the double coset space $W_{\mathcal{F}}(S)\backslash W/W_\Theta$. See [1]

On the other hand in [2] and [4] the maximal invariant chain transitive sets were described for general flows Φ on flag bundles, under the assumption that the flow on the base space is chain transitive. The approach consists in shadowing the flow by semigroups of homeomorphisms to take advantage of nice properties of the semigroup actions on flag manifolds.

From these results there exists a subgroup $W(\Phi)$ such that the number of maximal invariant chain transitive sets on the maximal flag bundle is in bijection with $W(\Phi)\backslash W$ while in a flag bundle with fiber B_Θ the chain transitive sets are parametrized by $W(\Phi)\backslash W/W_\Theta$, see [2]

In the sequel, we consider a transitive invariant control system Σ on a semisimple Lie group G, i.e., the Lie algebra generated by the invariant vector fields X, Y^j, $j = 1, \cdots, m$ coincide with $\mathbf{g}$. Then, we get

Theorem 3 *For the control flow Φ induced by Σ on the fiber bundle $\mathcal{U} \times G$ it holds*

$$W_{\mathcal{F}}(S) = W(\Phi).$$

In this context, the semigroup of the system is given by

$$S = \bigcup_{t \geq 0} A(t),$$

where the attainable set from the identity at time t, is $A(t) = \{\pi(1, u, t) : u \in \mathcal{U}\}$. Furthermore, the family $\mathcal{F}$ of subsets of S is defined by

$$\mathcal{F} = \{\bigcup_{t > T} A(t) : T \geq 0\}.$$

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