A NUMERICAL APPROACH FOR SOLVING OPTIMAL CONTROL OF HYBRID SYSTEMS*

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Abstract. This study deals with a numerical method for solving some control problems governed by hybrid dynamical systems. Our approach is based on the Hamilton-Jacobi approach.

Main results. The term hybrid system refers to a general framework that can be used to model a large class of systems that arise whenever a collection continuous- and discrete-time dynamics are put together in a single model [**?**, **?**, **?**]. Consider the controlled system: $\dot{X}(t) = f(X(t), q(t), u(t))$, with an initial condition $X(0) = x$, and where $x \in \mathbb{R}^d$, X is the continuous state and $q \in \mathbb{I}$ is the discrete one. The dynamics $f : \Omega \times \mathbb{I} \times U \to \mathbb{R}^d$ is continuous and the control set $\mathcal{U}$ is the set of all measurable function $u : (0, \infty) \to U$, where U is a compact set of $\mathbb{R}^m$ $(m \geq 1)$. The trajectory undergoes discrete jump when it hits a predifined set A, the autonomous jump set and C, the controlled jump set of $\mathbb{R}^d$. More precisely,

- on hitting A, the trajectory can jump to a predefined set D and continue with another discrete state $q' \in \mathbb{I}$. This jump is given by prescribed transition map $g : \mathbb{R}^d \times A \times \mathcal{V} \to D \times \mathbb{I}$, where $\mathcal{V}$ is the discrete control set. We denote by τ_i an arrival time to A, and by $X(\tau_i^-, q(\tau_i^-))$ the point before a jump. The point after the jump and the new discrete state value will be denoted by $(X(\tau_i^+), q(\tau_i^+)) = g(X(\tau_i^-), w)$.
- When the trajectory evolves in the set C, the controller can choose either to jump or not to jump. If the controller chooses to jump, then the continuous trajectory is moved to a new point in D, and the discrete state takes another value in $\mathbb{I}$. By ξ_i we denote a switching time. The controlled jump destination of $(X(\xi_i^-, u()), q(\xi_i^-))$ is $(X(\xi_i^+), q(\xi_i^+)) \in D \times \mathbb{I}$.

For example, the dynamics for $\tau_i < \xi_k < \tau_{i+1}$ is given by:

$$\dot{X}(t) = f(X(t), q(t), u(t)) \qquad \tau_i < t < \xi_k \tag{0.1a}$$

$$(X(\tau_i^+), q(\tau_i^+) = g(X(\tau_i^-), q(\tau_i^-), w) = (x_i', q_i') \qquad \text{and} \tag{0.1b}$$

$$q(t) := q(\tau_i^+) \quad \tau_i < t < \xi_k \tag{0.1c}$$

$$\dot{X}(t) = f(X(t), q(t), u(t)) \qquad \xi_k < t < \tau_{i+1} \tag{0.1d}$$

$$(X(\xi_k^+), q(\xi_k^+) \in D \times \mathbb{I} \qquad \text{and } q(t) := q(\xi_k^+) \quad \xi_k < t < \tau_{i+1} \tag{0.1e}$$

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Classical assumptions will be made on the sets A, C, D and on the functions f and g. For every strategy $\theta := (u(\cdot), v(\cdot), (\xi_i), (\tau_k))$, we associate the cost defined by:

$$J(x,\theta) := \int_0^{+\infty} \ell(X(t), q(t), u(t)) e^{-\lambda t}\, dt + \sum_{k=0}^{\infty} C_a(X(\tau_k^-), v) e^{-\lambda \tau_k} + $$

$$+ \sum_{i=0}^{\infty} C_c(X(\xi_i^-), \xi_i^-) e^{-\lambda \xi_i} \tag{0.2}$$

where λ is the discount factor, $\ell : \Omega \times \mathbb{I} \times U \to \mathbb{R}_+$ is the runing cost, $C_a : A \times \mathbb{I} \times \mathcal{V} \to \mathbb{R}_+$ is the autonomous jump cost and $C_c : C \times \mathbb{I} \times D \to \mathbb{R}_+$ is the controlled jump cost. The value function V is then defined as:

$$V(x,q) := \inf_{\theta \in \mathcal{U} \times \mathcal{V} \times [0,\infty) \times D} J(x,\theta). \tag{0.3}$$

The functions ℓ, C_c and C_a are assumed to be nonnegative and Lipschitz continuous. Moreover, $C_a(x,v)$ and $C_c(x,x')$ are uniformly bounded from below by some $C' > 0$. By using some viscosity arguments as in [1], we prove that V is the unique bounded continuous solution of:

$$\lambda V(x,q) + H(x, D_x V(x,q)) = 0 \qquad \text{in } \mathbb{R}^n \setminus (A \cup C), \tag{0.4a}$$

$$\max(V(x,q) - \mathcal{N}V(x,q), \lambda V(x,q) + H(x,q,D_x V(x,q)) = 0 \qquad \text{in } C, \tag{0.4b}$$

$$V(x,q) - \mathcal{M}V(x,q) = 0 \qquad \text{in } A, \tag{0.4c}$$

where $H(x,q,p) := \sup_{u \in U}\{-\ell(x,q,u) - f(x,q,u)\cdot p\}$, $\mathcal{M}\phi(x,q) := \inf_{v\in\mathcal{V}}\{\phi(g(x,q,v)) + C_a(x,v)$ for any $x \in A$, and $\mathcal{N}\phi(x,q) := \inf_{x'\in D, q'\in\mathbb{I}}\{\phi(x',q') + C_c(x,x')\}$ for every $x \in C$.

We consider a family of space-time grids in the computational domain, indexed by the discretization parameter h. We consider schemes approximating (**??**) in fixed point form:

$$V^h(x,q) = T^h(x,q,V^h) = \begin{cases} M^h V^h(x,q) & \text{if } x \in A \\ \min\left\{N^h V^h(x,q), S^h(x,q,V^h)\right\} & \text{if } x \in C \\ S^h(x,q,V^h) & \text{else.} \end{cases} \tag{0.5}$$

Here, V^h is the numerical approximation of the value function V. We shall first discuss the convergence issues related to this class of approximation schemes, and discuss the reconstruction of optimal trajectories for the underlying control problem. Moreover, we will provide some numerical examples to show the efficiency of the proposed method.

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