STABILITY OF DISTRIBUTED POWER CONTROL ALGORITHMS FOR TIME-DEPENDENT WIRELESS NETWORKS

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EXTENDED ABSTRACT. We consider a general class of continuous-time, time-dependent, distributed power control algorithms for wireless networks. Using appropriately constructed Lyapunov functions, we show that any bounded power distribution resulting from these algorithms is uniformly asymptotically stable, i.e. deviations from this trajectory asymptotically converge to zero. Further, under mild restrictions on the system, we make use of Lyapunov–Razumikhin functions to show that, even when the system incorporates heterogeneous, time-varying delays, if there exists a power distribution along which the genaralized system nonlinearity is bounded, then this must also be uniformly asymptotically stable. Moreover, in both cases this stability is global, meaning that for all initial conditions the power distributions have the same asymptotic behavior.

Introduction. When designing a wireless network, a crucial factor to control is the power transmitted by individual nodes. This power must be sufficiently high to ensure that the node's connection remains reliable, but not so high as to cause interference with neighboring nodes or to have too large an effect upon battery life. Moreover, in large-scale networks control of the power transmission should be distributed, in the sense that the power control scheme should be based upon local interference and battery life measurements rather than requiring a central control system.

One important decentralized power control algorithm for wireless networks is the Foschini–Miljanic algorithm, introduced in [1] with a generalized discrete-time version studied in [4]. A generalized class of continuous-time algorithms was studied in [3], where it was proved that, even if delays are present in the system, any fixed point is necessarily uniformly asymptotically stable. We consider a further generalization of this system by allowing the system nonlinearity to depend explicitly on time. The proliferation of mobile wireless network use, where the link gains between users may vary with time as the users are in relative motion, makes the study of time-dependent wireless systems both relevant and important.

Undelayed system. We consider a wireless system consisting of N users. Let the transmitted power of user i at time t be given by $p_i(t)$, and define the power vector by $p := (p_1, p_2, \dots, p_N)^T$. Generalizing the approach of [3], we consider the system

$$\frac{dp_i(t)}{dt} = k_i \left(-p_i(t) + I_i(t, p)\right), \tag{0.1}$$

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where the k_i are positive constants and $I : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is required to satisfy the following properties at all times t for all power values $p \geq 0$:[1]

- *Monotonicity*: if $p \geq p'$, then $I(t,p) \geq I(t,p')$,
- *Scalability*: there exists a continuous function $\delta : (1,\infty) \to \mathbb{R}^+$ such that, for any $\alpha > 1$, $I_i(t,p) - \frac{1}{\alpha} I_i(t,\alpha p) \geq \delta(\alpha)$ for all $i \in \{1,2,\ldots,N\}$.

Using these properties, it can be shown that I is both continuous in p and strictly positive for $p \geq 0$. This allows us to prove, for both the undelayed system and the delayed system, that trajectories starting in the positive orthant will never leave the positive orthant. If we restrict our attention only to solutions through positive initial data, this fact permits the coordinate change $\pi_i = \log\left(\frac{p_i}{P_i}\right)$, mapping a particular choice of solution $p = P(t)$ to $\pi = 0$ and transforming (0.1) to

$$\frac{d\pi_i}{dt} = \frac{k_i}{P_i e^{\pi_i}} \left[I_i(t, \operatorname{diag}(e^{\pi_1}, e^{\pi_2}, \ldots, e^{\pi_N})P) - e^{\pi_i} I_i(t,P) \right]. \tag{0.2}$$

We then use the Lyapunov function $V(\pi) = \max_i |\pi_i|$ to prove the following result:

THEOREM 0.1. *Suppose that $p = P(t)$ is a bounded solution of equation* (0.1) *subject to the initial condition $P(0) > 0$. Then $P(t)$ is uniformly asymptotically stable.*[2]

REMARK. It is important to note that if $p(t)$ is any solution of (0.1) with some p_i initially zero, then the positivity of $I_i(0,0)$ means that p_i must be initially increasing. Therefore, $p_i(t) > 0$ for all $t > 0$, and the theorem may be applied starting at any positive initial time to take such trajectories into account. Consequently, this result applies to all trajectories with non-negative initial power values and there is no loss of generality.

The fact that $V(\pi)$ is radially unbounded means that our argument shows that $\pi = 0$ is globally uniformly asymptotically stable. In view of the above remark, this in fact means that *any* trajectory $p(t)$ with $p(0) \geq 0$ will tend uniformly to $P(t)$ as $t \to \infty$. This is stated in the theorem below.

THEOREM 0.2. *Suppose that $p = P(t)$ is a bounded solution of equation* (0.1) *subject to the initial condition $P(0) \geq 0$. Then any trajectory $p(t)$ with $p(0) \geq 0$ will tend uniformly to $P(t)$ as $t \to \infty$.*

Delayed system. We then incorporate delays into (0.1), allowing I to depend on not just the current power states, but also those at earlier times. Specifically, we consider the delayed system

$$\frac{dp_i(t)}{dt} = k_i \left(-p_i(t) + I_i(t, p^{d_i}(t)) \right), \tag{0.3}$$

where $p^{d_i}(t) = (p_1(t-\theta_{i1}(t)), p_2(t-\theta_{i2}(t)), \ldots, p_N(t-\theta_{iN}(t)))^T$, with all delays θ_{ij} restricted to lie in some interval $[0,r]$. We also assume the previously stated properties of I, and additionally require in the scalability condition that $\delta(\alpha)$ be non-decreasing.

As in the undelayed case, we substitute $\pi_i = \log\left(\frac{p_i}{P_i}\right)$, yielding the delayed equivalent of (0.2)

$$\frac{d\pi_i}{dt} = \frac{k_i}{P_i e^{\pi_i}} \left[I_i\left(t, \operatorname{diag}\left(e^{\pi_1^{d_i}}, e^{\pi_2^{d_i}}, \ldots, e^{\pi_N^{d_i}}\right) P^{d_i}\right) - e^{\pi_i} I_i\left(t, P^{d_i}\right) \right].$$

[1] By vector inequalities, such as $p \geq p'$ with $p, p' \in \mathbb{R}^N$, we mean component-wise inequality.

[2] A solution $P(t)$ of (0.1) is said to be uniformly asymptotically stable if, for all $t_0 \geq 0$, both of the following conditions are satisfied: (i) given any $\epsilon > 0$, there exists $\sigma > 0$, independent of t_0, such that $\|p(t_0) - P(t_0)\| < \sigma$ implies $\|p(t) - P(t)\| < \epsilon$ for all $t > t_0$, and (ii) there exists $c > 0$, independent of t_0, such that $\|p(t_0) - P(t_0)\| < c$ implies $p(t) - P(t) \to 0$ uniformly as $t \to \infty$.

In order to address the infinite-dimensional character of the problem, we use the methods of Razumikhin [2] to investigate the stability of $\pi = 0$. We introduce $V(\pi) = \max_i |\pi_i|$ as our candidate Lyapunov–Razumikhin function, and then prove the existence of continuous, positive, non-decreasing functions q and w such that $q(s) > s$ for all $s > 0$ and

$$\dot{V}(t, \pi(t)) \leq -w(|\pi(t)|) \text{ whenever } V(t+\theta, \pi(t+\theta)) \leq q(V(t, \pi(t)))$$

for $\theta \in [-r, 0]$. In this way, we prove the following theorem:

THEOREM 0.3. *Suppose that* $p = P(t)$ *is a solution of equation* (0.3) *satisfying* $P(\theta) > 0$ *for all* $\theta \in [-r, 0]$. *Suppose further that there exists a positive constant* C *such that* $I_i(t, P^{d_i}(t)) < C$ *for all* $i \in \{1, 2, \ldots, N\}$ *and all* $t \geq 0$. *Then* $P(t)$ *is uniformly asymptotically stable.*

REMARK. Similarly to the undelayed case, if $p(t)$ is a solution of (0.3) with initial data that is non-negative, then $p(t)$ is guaranteed to be positive for all $t > 0$, meaning that we may invoke this theorem starting from any initial time exceeding the maximum delay, r, to deal with such cases. Therefore, there is again no loss of generality since the result thus extends to all trajectories with non-negative initial data, provided that the stated conditions on P are satisfied starting from this new initial time.

The radial unboundedness of $V(\pi)$ and the implications of this remark mean that, provided a solution $P(t)$ satisfying all of the stated conditions exists, *any* trajectory $p(t)$ with non-negative initial data will tend uniformly to $P(t)$ as $t \to \infty$. This is summarized in the theorem below.

THEOREM 0.4. *Suppose that* $p = P(t)$ *is a solution of equation* (0.3) *satisfying* $P(\theta) \geq 0$ *for all* $\theta \in [-r, 0]$. *Suppose further that there exists a positive constant* C *such that* $I_i(t, P^{d_i}(t)) < C$ *for all* $i \in \{1, 2, \ldots, N\}$ *and all* $t \geq 0$. *Then any trajectory* $p(t)$ *with* $p(\theta) \geq 0$ *for all* $\theta \in [-r, 0]$ *will tend uniformly to* $P(t)$ *as* $t \to \infty$.

Conclusions. We conclude that any bounded solution of (0.1) subject to non-negative initial conditions is necessarily uniformly asymptotically stable. Further, even if the system includes heterogeneous, time-varying delays, as in (0.3), any solution along which the system nonlinearity is bounded is also uniformly asymptotically stable, subject to the stated conditions holding. Moreover, since the function V is radially unbounded in each case, the uniform stability is global in the sense that all trajectories in the non-negative orthant must converge uniformly to the particular trajectory considered, $p = P(t)$. We have produced several numerical simulations that illustrate this behavior.

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