REACHABILITY FOR STATE CONSTRAINED STOCHASTIC CONTROL PROBLEMS.

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EXTENDED ABSTRACT. This work is concerned with the reachability problem for stochastic controlled system in presence of state constraints. Given a system of controlled linear stochastic differential equations, our aim is to characterize the set of points from which it is almost surely possible to reach a target set satisfying the state constraints. The main idea of our approach is to connect the characterization of the constrained reachable set to an unconstrained control problem in order to avoid some regularity issues related to the presence of state constraints. A similar approach has been showed to be successful in the deterministic context leading to constructive and powerful way for computing numerically the reachable set. We present two different approaches. In each case a continuous value function is defined such that the reachable set coincides exactly with its zero level-set.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}$ and a Brownian motion $W(\cdot)$ in $\mathbb{R}^p$, we consider a system of controlled linear stochastic differential equations in $\mathbb{R}^d$

$$dX(s) = (AX(s) + Bu(s))ds + (CX(s) + Du(s))dW(s) \qquad s \in [0,T],\ T > 0, \quad (0.1)$$

where $u \in \mathcal{U}$, the set of progressively measurable processes from $[0,T]$ with values in a compact and convex set $U \subseteq \mathbb{R}^k$, A, B, C and D are matrices of opportune size and $X^u_{t,x}(\cdot) \in \mathbb{R}^d$ denotes the strong solution of (0.1) corresponding to the initial data $X(t) = x$. We also denote with $\mathcal{T} \subseteq \mathbb{R}^d$ and $\mathcal{K} \subseteq \mathbb{R}^d$ respectively the target set and the set of state-constraints, both are assumed to be convex sets.

Our aim is to characterize $\mathcal{R}_t$, defined as the set of points from which it is almost surely possible to reach the target $\mathcal{T}$ remaining in $\mathcal{K}$, that is the set of points $x \in \mathbb{R}^d$ such that

$$\exists u \in \mathcal{U}, \qquad \left(X^u_{t,x}(T) \in \mathcal{T} \text{ and } X^u_{t,x}(\theta) \in \mathcal{K}, \forall \theta \in [t,T] \right) \qquad \text{a.s.} \qquad (0.2)$$

Control problems in presence of state constraints have been considered by many authors because there is a growing interest of this kind of problems in many applications, see for instance [4]. In the case of $\mathcal{T} = \mathbb{R}^d$ (no target constraint), the set $\mathcal{R}_t$ is equivalent to a stochastic viability set, which has been studied in [5] (in the case of an infinite horizon).

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We introduce here a new idea, based on a level-set approach, for characterizing the set $\mathcal{R}_t$. The main feature of our approach is that it connects the constrained set $\mathcal{R}_t$ to an unconstrained control problem, in order to avoid some regularity issues related to the presence of state constraints. A similar approach has been showed to be successful in the deterministic case, see [2], leading to constructive and powerful way for computing numerically the set $\mathcal{R}_t$.

In the present work we follow two different approaches (detailed results can be found in [3]). In each case a *continuous* auxiliary value function $v(t,x)$ is defined such that the following equivalence holds

$$v(t,x)=0 \Longleftrightarrow x\in\mathcal{R}_t. \tag{0.3}$$

We obtain first a Dynamic Programming Principle (DPP), and then a characterization in terms of a second order Hamilton-Jacobi-Bellman (HJB) equation.

Our first approach is linked to a classical stochastic optimal control problem. We define the following standard value function

$$v_1(t,x)=\inf_{u\in\mathcal{U}}\mathbb{E}\left[\varphi(X^u_{t,x}(T))+\int_t^T g(X^u_{t,x}(\theta))d\theta|\mathcal{F}_t\right], \tag{0.4}$$

and in that case, both the DPP and the HJB equation are deduced by classical arguments. In order to obtain the equivalence (0.3), the functions φ and g involved in the definition of the value function can be choosen to be the distance functions respectively to $\mathcal{T}$ and $\mathcal{K}$.

For the second approach we define the following value function

$$v_2(t,x)=\inf_{u\in\mathcal{U}}\mathbb{E}\left[\varphi(X^u_{t,x}(T))\bigvee\max_{\theta\in[t,T]}g(X^u_{t,x}(\theta))|\mathcal{F}_t\right], \tag{0.5}$$

where $a\vee b:=\max(a,b)$. We prove again that, for a suitable choice of φ and g, the assertion (0.3) holds. However, the function v_2 doesn't satisfy any DPP. For this reason, we introduce another auxiliary control problem:

$$w(t,x,y)=\inf_{u\in\mathcal{U}}\mathbb{E}\left[\varphi(X^u_{t,x}(T))\bigvee\max_{\theta\in[t,T]}g(X^u_{t,x}(\theta))\ \bigvee\ y\ |\ \mathcal{F}_t\right]. \tag{0.6}$$

The value function w satisfies a DPP which seems to be new, and, as a consequence, an HJB equation. Then a characterization of the set $\mathcal{R}_t$ follows for a suitable choice of φ and g.

Control problems with a maximum cost have been involved in the deterministic case for dealing with state constraints in absence of controllability assumptions, see [2]. In the stochastic context, if $g(x)\equiv x$ and $\varphi\equiv 0$, the formulation (0.5) has been used in [1] for modelling the price of lookback options in finance. The arguments developed in [1] cannot be generalized to the setting (0.5) and (0.6).

Here, we propose a new approach to derive the DPP and we give the precise HJB equation that characterizes the function w. Our two approaches lead to new ways for dealing with state constraints in a stochastic context and this theoretical characterization will be showed to be particularly useful also for numerical purposes. Further studies will concern more general state-constrained stochastic control problems with nonlinear dynamics.

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