INTRODUCING OPEN ERGODIC THEORY*

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EXTENDED ABSTRACT. We introduce a formalism of open stochastic systems. This is done by an extension of the language of ergodic theory, which was initially designed to describe autonomous systems with a random initial condition. Our formalism can be seen as an adaptation of Willems's behavioural framework to stochastic systems. We show how to define the capacity and entropy rate of a system, as a generalisation of the usual capacity found in communication theory and entropy rate defined in ergodic theory and provide prospects of applications.

1. Introduction. In this contribution we propose a formalism aiming at unifying ergodic theory and open systems theory.

Ergodic theory [3] is concerned with transformations of the kind $f : X \to X$ of a state space X endowed with a probability measure left invariant by the transformation. This transformation creates trajectories $x_{t+1} = f(x_t)$ determined by the initial condition. In other words, ergodic theory studies deterministic autonomous systems with a random initial condition. This theory initiated by Poincaré, Hadamard, Morse and others has reached a great maturity. In particular Kolmogorov, Sinai, Ornstein, etc. have rapidly drawn the lessons of information theory, leading to a characterisation of such systems with entropy rates. Much of ergodic theory relies on symbolic dynamics, the main idea of which goes as follows. We define an observable $h : X \to Y$ and consider the set of all possible observed trajectories (which is a subset of $Y^{\mathbb{Z}}$). This set forms a stochastic process, whose probability of events are determined by the probability measure on the initial state x_0. The study of this stochastic process allows to characterise the original dynamics. Often Y is chosen to be a finite set of symbols, hence the name of the technique.

On the other hand, open systems theory studies systems of the kind $x_{t+1} = f(x_t) + g(u_t); y_t = h(x_t) + i(u_t)$, where u_t is seen as an input signal, imposed by the environment. The input may be seen as a random disturbance or chosen by an engineer in order to steer the state trajectory $(x_t)_{t\in\mathbb{Z}}$ or the output trajectory $(y_t)_{t\in\mathbb{Z}}$. An alternative formulation has been proposed by Jan Willems [5], where the central object is the behaviour, i.e., the set of all trajectories $(u_t, y_t)_{t\in\mathbb{Z}}$ that can be observed.

This short and obviously simplistic account is aimed at highlighting the analogies between open systems theory and ergodic theory. Yet, the two theories have been

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developing rather independently. In particular the articulation with information theory and entropy rates has not been studied until the recent pioneering work [2, 1] proposing to measure the difficulty to control a system or maintain it invariant by an entropy. It is our goal here to propose a single class of objects containing open systems, autonomous ergodic systems and communication channels. This will allow a systematic exploitation of information theory and ergodic theory into open systems theory, resulting hopefully in a better capture of the interaction of systems and communication channels.

2. Open stochastic systems. As pointed out in the introduction, the main tool of ergodic theory (symbolic dynamics) is strongly related to the behavioural approach to open systems. The theory of behaviours has been much developed in the deterministic case, and recently Willems proposed an elegant extension for stochastic systems [4].

We generalise this approach to the following definition. Consider an observable variable z taking values in the set Z, and endow the observable trajectory set $Z^{\mathbb{Z}}$ with a suitable σ-algebra of events. An open stochastic system is a family of stochastic processes taking place in $Z^{\mathbb{Z}}$. If the system is autonomous, then this family is a unique stochastic process determined by the probability measure on the initial state. If the system is open, then we may see z as (u, y), and every admissible stochastic law on the input u_t induces a stochastic process on z_t. Communication channels fall into that scope. For example a memoryless binary symmetric channel can be easily expressed as an open stochastic system, for $Z = U \times Y = \{0, 1\} \times \{0, 1\}$.

Interconnection is defined similarly to deterministic behaviour theory, as an intersection.

3. Capacity. We may swiftly generalise the notion of capacity, from communication theory to general open systems. Consider two partial observables u and y deduced from the observable z, defined by the projections $\pi_U : Z \to U$ and $\pi_Y : Z \to Y$. Those variables may be thought as an input and an output, for example. Every stochastic process on z_t induces two dependent processes on u_t and y_t, and we may compute the mutual information rate between these stochastic processes. The supremum of this mutual information rate is called the capacity between u and y for the system.

This coincides with the usual notion of capacity when the open system is a communication channel. For a linear system of the form $x_{t+1} = Ax_t + Bu_t$ and $y_t = Cx_t + Du_t$, we prove that the capacity is $\sum_i |\log|\lambda_i||$, where λ_i is a nonzero eigenvalue of A.

4. Entropy rate. In the above, we assume any possible stochastic process for the input. We may however restrict it to the possible input processes that are provided by a deterministic feedback controller connected to the open system. We can characterise such processes intrinsically by a causality-like condition on (u_t, y_t). In this case the capacity for a linear system is lowered to $\sum_{i:|\lambda_i|>1} |\log|\lambda_i||$. We call it causal entropy rate. Reverting the causality condition, we obtain an anticausal entropy rate of $\sum_{i:|\lambda_1|>1} \log|\lambda_i|$. We therefore decompose the capacity as a sum of a causal and anticausal part. As we see in this case, the causal entropy rate characterises the unstable part of the dynamics while the stable part is reflected by the anticausal part.

The causal entropy rate is conjecturely related to the entropy rates defined in the control literature [2, 1] through a variational theorem similar to the variational theorem found in ergodic theory.

5. Networks. Our future goal, for which much remains to be done, is to compute bounds on global capacities and entropy rates for arbitrary interconnections of open systems (including communication channels), in terms of individual entropy rates and capacities, in order to evaluate in a general way how instabilities are tamed or enhanced when put together.

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