Predictive Path Following without Terminal Constraints

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Nonlinear model predictive control (NMPC) for set point stabilization has been intensively investigated. Furthermore, by now several formulations for non-classical stabilization have been proposed, such as economic NMPC, and NMPC for trajectory-tracking or path-following problems. While in trajectory tracking the objective is to track an explicitly time dependent reference, in path following a geometric reference without any preassigned timing information should be followed. Often path-following problems are solved without an explicit consideration of constraints, e.g. by means of backstepping techniques. Here, we propose an approach based on a tailored NMPC scheme which allows to consider input constraints explicitly. Existing NMPC approaches for path following use terminal constraints and end penalties, or contraction constraints to derive path convergence, see [1, 2, 4]. These constraints, however, might lead to a high computational burden. In contrast to these works we provide sufficient conditions which guarantee path convergence without such constraints for exactly feedback linearizable systems. We rely on recent results that guarantee stability of discrete time NMPC via controllability assumptions [3] and related first steps towards a continuous time extension [5].

Problem Statement and Proposed Control Strategy

Specifically, we consider square input affine MIMO systems

$$\dot{x} = f(x) + g(x)u, \quad x(0) = x_0 \in \mathcal{X}_0 \tag{1a}$$

$$y = h(x) \tag{1b}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathcal{U} \subset \mathbb{R}^m$ and $y \in \mathbb{R}^m$ refer to inputs and outputs, respectively. The inputs $u : \mathbb{R} \to \mathcal{U}$ are piecewise continuous and take values

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in a compact set $\mathcal{U}$. Briefly this is denoted as $u(\cdot) \in \mathcal{PC}(\mathcal{U})$. The maps f, g, h are assumed to be sufficiently often continuously differentiable and (1a) is locally Lipschitz. Additionally, we make the following assumption.

Assumption 1 (System properties).

(i) *For a sufficiently large simply connected set $\mathcal{X} \subseteq \mathbb{R}^n$ (1) has a well-defined vector relative degree $r = (r_1, \ldots, r_m)$ with $\sum_{i=1}^{m} r_i = n$.*

(ii) *For $u = 0$ the origin $x = 0$ is the only steady state of (1) contained in $\{x \mid h(x) = 0\}$.*

This assumption means that we consider systems, which are exactly feedback linearizable in a sufficiently large subset of the state space $\mathbb{R}^n$. This also implies that (1a) has no internal dynamics with respect to the considered output (1b).
The objective is to follow an a priori known path $\mathcal{P} \subset \mathbb{R}^m$, which is given as a regular curve

$$\mathcal{P} = \{y \in \mathbb{R}^m \mid \theta \in \mathbb{R} \mapsto p(\theta)\}\,. \tag{2}$$

Here $\theta \in \mathbb{R}$ is called the path parameter. And $p(\theta)$ is a parametrization of $\mathcal{P}$, which is assumed to be sufficiently often continuously differentiable and $p(0) = 0$. The path is a geometric reference without an explicit requirement *when to be where* on $\mathcal{P}$. The conceptual idea is to treat the path parameter θ as a virtual state whereby the evolution of θ can be influenced by an extra input. Thus the path parameter dynamics $t \mapsto \theta(t)$ are described by a *timing law*, which is a degree of freedom in the controller design. We rely on the timing law $\theta^{(\hat{r}+1)} = v$ where $\hat{r} = \max_i\{r_i\}$ is the largest element of the vector relative degree of (1) and $v(\cdot) \in \mathcal{PC}(\mathcal{V}), \mathcal{V} \subset \mathbb{R}$. Using $z := (\theta, \dot{\theta}, \ldots, \theta^{(\hat{r})})^T$ this dynamic timing law can be expressed as

$$\dot{z} = Az + Bv, \quad z(0) = (\theta_0, 0, \ldots, 0)^T. \tag{3}$$

Now, we are ready to state the path-following problem in a formal way.

Problem 1 (Input constrained path following).
Given system (1) and the path $\mathcal{P}$ from (2) design a controller which achieves:

(i) *Convergence towards the path: $\lim\limits_{t\to\infty} \|h(x(t)) - p(\theta(t))\| = 0$.*

(ii) *Convergence on the path: $\lim\limits_{t\to\infty} \|\theta(t)\| = 0$.*

(iii) *Constraint satisfaction: The input constraints $u(t) \in \mathcal{U}$ are satisfied and the states $x(t)$ remain bounded.*

The challenge is to obtain the real system inputs $u(t)$ as well as the evolution of the reference, which is defined by $t \mapsto \theta(t)$, at once while satisfying the constraints. In essence, the problem can be understood as an online trajectory generation on the 1-dimensional manifold $\mathcal{P}$. However, note that the system initial condition x_0, in general, does not lie on the path, i.e. $h(x_0) \notin \mathcal{P}$. Furthermore, note that part (ii) implies to stop at the final path point $\theta = 0$.
We propose to solve this problem via a tailored NMPC scheme, i.e. we want to compute $u(t)$ as well as the timing $\theta(t)$ by repetitive solution of an optimal control

problem (OCP). As standard in NMPC this optimization is done at each sampling instant $t_k = k\delta, k \in \mathbb{N}, \delta > 0$. Predicted system states, outputs, and inputs are denoted by superscript $\bar{\cdot}$. Furthermore, optimal inputs are denoted with superscript $\cdot^{\star}$. The functional to be minimized at each sampling instant is given by

$$J(t_k, x(t_k), \bar{u}(\cdot), \bar{v}(\cdot)) = \int_{t_k}^{t_k+T_p} F\left(\bar{e}(\tau), \bar{\theta}(\tau), \bar{u}(\tau), \bar{v}(\tau)\right) d\tau, \tag{4a}$$

where $T_p > \delta$ is the prediction horizon. The cost function $F : \mathbb{R}^m \times \mathbb{R} \times \mathcal{U} \times \mathbb{R} \to \mathbb{R}_0^+$ penalizes the path error $e = h(x) - p(z_1)$, the distance to the final path point ($\theta = 0$), and the inputs u, v. It is lower bounded by a class $\mathcal{K}$ function $\alpha(\|(e, \theta, u, v)\|)$. The OCP to be solved repetitively is as follows:

$$\underset{(\bar{u}(\cdot),\bar{v}(\cdot))\in\mathcal{PC}(\mathcal{U}\times\mathcal{V})}{\text{minimize}} \quad J(t_k, x(t_k), \bar{u}(\cdot), \bar{v}(\cdot)) \tag{4b}$$

$$\text{subject to} \quad \begin{pmatrix} \dot{\bar{x}} \\ \dot{\bar{z}} \end{pmatrix} = \begin{pmatrix} f(\bar{x}) \\ A\bar{z} \end{pmatrix} + \begin{pmatrix} g(\bar{x})\bar{u} \\ B\bar{v} \end{pmatrix}, \quad \begin{array}{c} \bar{x}(t_k) = x(t_k) \\ \bar{z}(t_k) = \bar{z}(t_k, \bar{z}(t_{k-1})|v) \end{array} \tag{4c}$$

$$\bar{e} = h(\bar{x}) - p(\bar{z}_1) \tag{4d}$$

$$\bar{\theta} = \bar{z}_1 \tag{4e}$$

$$\forall t \in [t_k, t_k + T_p] : \quad \bar{u}(t) \in \mathcal{U}, \quad \bar{v}(t) \in \mathcal{V}. \tag{4f}$$

Note that the system used for prediction (4c) is an augmented one, which is composed by the system to be controlled (1) and the path parameter dynamics (3). The outputs (4d-e) refer to the path error and the path parameter, respectively. As usual in NMPC the measured state information $x(t_k)$ serves as initial condition in (4c). The initial condition for the virtual state z is taken from the last predicted trajectory, i.e. $\bar{z}(t_k) = \bar{z}(t_k, \bar{z}(t_{k-1})|v(\cdot))$. The input applied to the real system (1) is obtained in a receding horizon fashion, i.e. for $t \in [t_k, t_k + \delta) : \ u(t) = \bar{u}^{\star}(t, x(t_k))$.

Naturally, the question arises under which conditions the proposed NMPC scheme guarantees convergence to the path $\mathcal{P}$. To verify that the proposed control scheme solves the path-following problem we rely on a controllability assumption, similar to results on NMPC for stabilization of discrete and continuous time systems, cf. [3, 5].

Assumption 2 (Controllability).
For all $(x_0, z_0) \in \mathbb{R}^n \times \mathbb{R}^{\hat{r}+1}$ there exist controls $(u(\cdot), v(\cdot))^T \in \mathcal{PC}(\mathcal{U} \times \mathcal{V})$ defined on $[0, \infty)$ such that the state trajectories $x(t, x_0|u(\cdot))$, $z(t, z_0|v(\cdot))$ generate output trajectories $e(t), \theta(t)$ via (4d-e) which satisfy

$$\forall T \geq 0 : \quad \int_0^T F\left(e(t), \theta(t), v(t), u(t)\right) dt \ \leq \ B(T) \min_{(u,v)\in\mathcal{U}\times\mathcal{V}} F(e_0, \theta_0, u, v) \tag{5}$$

where the function $B : \mathbb{R}^+ \to \mathbb{R}^+$ is $\mathcal{C}^1$, non-decreasing, and bounded.

Based on this the following result can be derived.

Proposition 1.
Suppose Assumptions 1 & 2 hold, then a finite prediction horizon $T_p \in (\delta, \infty)$ exists such that the NMPC scheme (4) *solves Problem 1.*

Proof. Due to space limitations we provide only a brief sketch of the proof. Compared to standard NMPC schemes for stabilization the main challenge is that the cost function F is only positive semi-definite with respect to the augmented state (x, z). Firstly, note that part (i) of Assumption 1 guarantees that the augmented system (4c) has a well-defined vector relative degree $r = (r_1, \ldots, r_m, \hat{r} + 1)$ and hence no internal dynamics with respect to the output (e, θ), see [1]. In essence, Assumption 1 ensures that the augmented system (4c) is 0-detectable. In other words, whenever the output (e, θ) and the input (u, v) of the augmented system (4c) converge, i.e. $\lim_{t\to\infty}(e, \theta) = 0$ and $\lim_{t\to\infty}(u, v) = 0$, then the state (x, z) also converges.
Secondly, one relies on a continuous time version of suboptimality estimates for NMPC schemes, cf. [3]. Using the ideas presented in [5], and based on Assumption 2, one can show that for a given sampling time $\delta > 0$ a sufficiently large $T_p \in (\delta, \infty)$ ensures convergence of the augmented output (e, θ) and the augmented input (u, v) to the origin. $\square$

Conclusions and Outlook

In this note we outline a conceptual framework for predictive path following of exactly feedback linearizable systems in the presence of input constraints. We sketch sufficient convergence conditions based on an augmented system description. In contrast to previous works [1, 2, 4] we do not rely on contraction or terminal constraints.
Note that the inclusion of state constraints is non-trivial. The main challenge is guaranteeing recursive feasibility of the OCP (4). Thus the consideration of path-following specific state constraints related to the path parameter dynamics (3)—e.g. restriction of the path parameter to an interval ($\theta = z_1 \in [\theta_0, 0]$), or forward motion constraints ($\dot{\theta} = z_2 \geq 0$)—is subject of future work.

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