A GENERALIZED FRACTIONAL CALCULUS OF VARIATIONS WITH APPLICATIONS*

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Key words. Generalized fractional variational analysis, Euler–Lagrange equations, natural boundary conditions, fractional action-like variational approach, Caldirola–Kanai dynamical systems.

AMS subject classifications. 49K05, 49K21, 26A33.

EXTENDED ABSTRACT. In this work we study fractional variational problems defined using a generalized fractional integral and with Lagrangians depending on the classical derivative, generalized fractional integrals and derivatives. We obtain necessary optimality conditions for the basic and isoperimetric problems, as well as transversality conditions for free boundary value problems. As applications the generalized fractional action-like variational problems are discussed.

We start with definitions of generalized fractional operators, which were introduced in [1] and then later discussed in [9, 8]. As particular cases, by choosing appropriate kernels, these operators are reduced to standard fractional integrals and fractional derivatives (see, e.g., [5]).

DEFINITION 0.1 (Generalized fractional integral). *The operator K_P^α is given by*

$$K_P^\alpha\left[f\right](x) = K_P^\alpha\left[t \mapsto f(t)\right](x) = p\int_a^x k_\alpha(x,t)f(t)dt + q\int_x^b k_\alpha(t,x)f(t)dt,$$

where $P = \langle a, x, b, p, q\rangle$ is the parameter set *(p-set for brevity), $x \in [a,b]$, p, q are real numbers, and $k_\alpha(x,t)$ is a kernel which may depend on α. The operator K_P^α is referred as the* operator K *(K-op for simplicity) of order α and p-set P, while $K_P^\alpha[f]$ is called the* operation K *(or K-opn) of f of order α and p-set P.*

The generalized fractional derivatives A_P^α and B_P^α are defined in terms of the generalized fractional integral K-op.

DEFINITION 0.2 (Generalized Riemann–Liouville fractional derivative). *Let P be a given parameter set and $0 < \alpha < 1$. The operator A_P^α is defined by $A_P^\alpha = D \circ K_P^{1-\alpha}$, where D denotes the standard derivative operator, and is referred as the* operator A *(A-op) of order α and p-set P, while $A_P^\alpha[f]$, for a function f such that $K_P^{1-\alpha}[f] \in AC([a,b])$, is called the* operation A *(A-opn) of f of order α and p-set P.*

DEFINITION 0.3 (Generalized Caputo fractional derivative). *Let P be a given parameter set and $\alpha \in (0,1)$. The operator B_P^α is defined by $B_P^\alpha = K_P^{1-\alpha} \circ D$, where D denotes the standard derivative operator, and is referred as the* operator B *(B-op) of order α and p-set P, while $B_P^\alpha[f]$, for a function $f \in AC([a,b])$, is called the* operation B *(B-opn) of f of order α and p-set P.*

*Part of the first author's Ph.D., which is carried out at the University of Aveiro under the Doctoral Programme *Mathematics and Applications* of Universities of Aveiro and Minho.

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A theory of the fractional calculus of variations started in 1996 with works of Riewe [10], in order to better describe nonconservative systems in mechanics. After that several approaches have been developed to generalize the least action principle and the Euler–Lagrange equations to include fractional derivatives [2, 6, 7, 9]. Here we study incommensurate fractional variational problems with generalized fractional operators.

Let us consider the problem of finding a function $y = t \mapsto y(t)$, $t \in [a,b]$, that gives an extremum (minimum or maximum) to the functional $\mathcal{J}(y)$ defined by

$$K_P^\alpha \left[t \mapsto F\left(t, y(t), y'(t), B_{P_1}^{\beta_1}[y](t), \ldots, B_{P_n}^{\beta_n}[y](t), K_{R_1}^{\gamma_1}[y](t), \ldots, K_{R_n}^{\gamma_n}[y](t)\right)\right](b) \tag{1}$$

when subject to the boundary conditions

$$y(a) = y_a, \quad y(b) = y_b, \tag{2}$$

where $\alpha, \beta_i, \gamma_i \in (0,1)$, $P =< a, b, b, 1, 0 >$, $P_i =< a, t, b, p_i, q_i >$, $R_i =< a, t, b, r_i, s_i >$, $i = 1, \ldots, n$. For simplicity of notation we introduce the operator $\{\cdot\}_{P_D,R_I}^{\beta,\gamma}$ defined by

$$\{y\}_{P_D,R_I}^{\beta,\gamma}(t) = \left(t, y(t), y'(t), B_{P_D}^{\beta}[\tau \mapsto y(\tau)](t), K_{R_I}^{\gamma}[\tau \mapsto y(\tau)](t)\right),$$

where

$$B_{P_D}^{\beta} := \left(B_{P_1}^{\beta_1}, \ldots, B_{P_n}^{\beta_n}\right), \ K_{R_I}^{\gamma} := \left(K_{R_1}^{\gamma_1}, \ldots, K_{R_n}^{\gamma_n}\right).$$

The operator K_P^α has kernel $k_\alpha(x,t)$, and for $i = 1, \ldots, n$ operators $B_{P_i}^{\beta_i}$ and $K_{R_i}^{\gamma_i}$ have kernels $h_{1-\beta_i}(t,\tau)$ and $h_{\gamma_i}(t,\tau)$, respectively. In the sequel we assume that:

(H1) Lagrangian $F \in C^1\left([a,b] \times \mathbb{R}^{2n+2}; R\right)$;

(H2) for $i = 1, \ldots, n$ functions $A_{P_i^*}^{\beta_i}\left[\tau \mapsto k_\alpha(b,\tau)\partial_{i+3}F\{y\}_{P_D,R_I}^{\beta,\gamma}(\tau)\right]$, $K_{R_i^*}^{\gamma}\left[\tau \mapsto k_\alpha(b,\tau)\partial_{n+3+i}F\{y\}_{P_D,R_I}^{\beta,\gamma}(\tau)\right]$, $D\left[t \mapsto \partial_3 F\{y\}_{P_D,R_I}^{\beta,\gamma}(t)k_\alpha(b,t)\right]$ and $t \mapsto k_\alpha(b,t)\partial_2 F\{y\}_{P_D,R_I}^{\beta,\gamma}(t)$ are continuous on (a,b);

(H3) functions $t \mapsto \partial_3 F\{y\}_{P_D,R_I}^{\beta,\gamma}(t)k_\alpha(b,t)$, $K_{P_i^*}^{1-\beta_i}\left[\tau \mapsto k_\alpha(b,\tau)\partial_{i+3}F\{y\}_{P_D,R_I}^{\beta,\gamma}(\tau)\right] \in AC([a,b])$, $i = 1, \ldots, n$;

(H4) kernels $k_\alpha(x,t)$, $h_{1-\beta_i}(t,\tau)$ and $h_{\gamma_i}(t,\tau)$, $i = 1, \ldots, n$, are such that we are in conditions to use Theorems 9, 10 and 11 of [9].

Theorem 0.4. *If y is a solution to problem (1)–(2), then y satisfies the generalized Euler–Lagrange equation*

$$\begin{aligned} k_\alpha(b,t)\partial_2 F\{y\}_{P_D,R_I}^{\beta,\gamma}(t) &- \frac{d}{dt}\left(\partial_3 F\{y\}_{P_D,R_I}^{\beta,\gamma}(t)k_\alpha(b,t)\right) \\ &- \sum_{i=1}^{n} A_{P_i^*}^{\beta_i}\left[\tau \mapsto k_\alpha(b,\tau)\partial_{i+3}F\{y\}_{P_D,R_I}^{\beta,\gamma}(\tau)\right](t) \\ &+ \sum_{i=1}^{n} K_{R_i^*}^{\gamma_i}\left[\tau \mapsto k_\alpha(b,\tau)\partial_{n+i+3}F\{y\}_{P_D,R_I}^{\beta,\gamma}(\tau)\right](t) = 0 \end{aligned} \tag{3}$$

for all $t \in (a,b)$.

Assume now that in problem (1)–(2) the boundary conditions (2) are substituted by

$$y(a) \text{ is free and } y(b) = y_b. \tag{4}$$

THEOREM 0.5. *If y is a solution to the problem of extremizing functional (1) with (4) as boundary conditions, then y satisfies the Euler–Lagrange equation (3). Moreover, the extra natural boundary condition*

$$\partial_3 F\left\{y\right\}^{\beta,\gamma}_{P_D,R_I}(a)k_\alpha(b,a)+\sum_{i=1}^{n}K^{1-\beta_i}_{P_i^*}\left[\tau\mapsto\partial_{i+3}F\left\{y\right\}^{\beta,\gamma}_{P_D,R_I}(\tau)k_\alpha(b,\tau)\right](a)=0 \quad (5)$$

holds.

Let us consider now the isoperimetric problem: among all functions $y:[a,b]\to\mathbb{R}$ satisfying boundary conditions (2) and an isoperimetric constraint of the form

$$\mathcal{I}(y)=K^\alpha_P\left[G\left\{y\right\}^{\beta,\gamma}_{P_D,R_I}\right](b)=\xi, \quad (6)$$

where $\xi\in\mathbb{R}$ is fixed, we look for the one that extremizes (i.e., minimizes or maximizes) functional (1). We assume that functional (6) satisfies hypotheses (H1)–(H4).

DEFINITION 0.6. *An admissible function $y\in C^1([a,b],\mathbb{R})$ is said to be an* extremal *for $\mathcal{I}$ if it satisfies the Euler–Lagrange equation (3) associated with functional in (6).*

THEOREM 0.7. *If y is a solution to the isoperimetric problem defined by (1)–(2), (6), and is not an extremal for $\mathcal{I}$, then there exists a real constant λ such that*

$$\begin{aligned}k_\alpha(b,t)\partial_2 H\left\{y\right\}^{\beta,\gamma}_{P_D,R_I}(t)&-\frac{d}{dt}\left(\partial_3 H\left\{y\right\}^{\beta,\gamma}_{P_D,R_I}(t)k_\alpha(b,t)\right)\\ &-\sum_{i=1}^{n}A^{\beta_i}_{P_i^*}\left[\tau\mapsto k_\alpha(b,\tau)\partial_{i+3}H\left\{y\right\}^{\beta,\gamma}_{P_D,R_I}(\tau)\right](t)\\ &+\sum_{i=1}^{n}K^{\gamma_i}_{R_i^*}\left[\tau\mapsto k_\alpha(b,\tau)\partial_5 H\left\{y\right\}^{\beta,\gamma}_{P_D,R_I}(\tau)\right](t)=0 \quad (7)\end{aligned}$$

for all $t\in(a,b)$, where $H(t,y,u,v,w)=F(t,y,u,v,w)-\lambda G(t,y,u,v,w)$ and $P_j^*=<a,t,b,q_j,p_j>$.

As an application, we extend some of the recent results of [3, 4], where the fractional action-like variational approach (FALVA) was proposed to model dynamical systems. FALVA functionals are particular cases of (1), when it does not depend on B-op and K-op. The Caldirola–Canai Lagrangians, that describe dynamical oscillatory systems with exponentially increasing time dependent mass are considered: for details see [9].

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