A GENERALIZED FRACTIONAL CALCULUS OF VARIATIONS
WITH APPLICATIONS

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EXTENDED ABSTRACT. In this work we study fractional variational problems defined using a generalized fractional integral and with Lagrangians depending on the classical derivative, generalized fractional integrals and derivatives. We obtain necessary optimality conditions for the basic and isoperimetric problems, as well as transversality conditions for free boundary value problems. As applications the generalized fractional action-like variational problems are discussed.

We start with definitions of generalized fractional operators, which were introduced in [1] and then later discussed in [9, 8]. As particular cases, by choosing appropriate kernels, these operators are reduced to standard fractional integrals and fractional derivatives (see, e.g., [5]).

Definition 0.1 (Generalized fractional integral). The operator $K_\alpha^p$ is given by

$$K_\alpha^p[f](x) = K_\alpha^p[t \mapsto f(t)](x) = p \int_0^x k_\alpha(x,t)f(t)dt + q \int_2^b k_\alpha(t,x)f(t)dt,$$

where $P = \{a,x,b,p,q\}$ is the parameter set ($p$-set for brevity), $x \in [a,b]$, $p,q$ are real numbers, and $k_\alpha(x,t)$ is a kernel which may depend on $\alpha$. The operator $K_\alpha^p$ is referred as the operator $K$ ($K$-op for simplicity) of order $\alpha$ and $p$-set $P$, while $K_\alpha^{p,p}$ is called the operation $K$ ($K$-op) of $f$ of order $\alpha$ and $p$-set $P$.

The generalized fractional derivatives $A_\alpha^p$ and $B_\alpha^p$ are defined in terms of the generalized fractional integral $K_\alpha^p$.

Definition 0.2 (Generalized Riemann–Liouville fractional derivative). Let $P$ be a given parameter set and $0 < \alpha < 1$. The operator $A_\alpha^p$ is defined by $A_\alpha^p = D \circ K_\alpha^{1,\alpha}$, where $D$ denotes the standard derivative operator, and is referred as the operator $A$ ($A$-op) of order $\alpha$ and $p$-set $P$, while $A_\alpha^{p,p}$, for a function $f$ such that $K_\alpha^{1,\alpha}[f] \in AC([a,b])$, is called the operation $A$ ($A$-op) of $f$ of order $\alpha$ and $p$-set $P$.

Definition 0.3 (Generalized Caputo fractional derivative). Let $P$ be a given parameter set and $\alpha \in (0,1)$. The operator $B_\alpha^p$ is defined by $B_\alpha^p = K_\alpha^{1-\alpha} \circ D$, where $D$ denotes the standard derivative operator, and is referred as the operator $B$ ($B$-op) of order $\alpha$ and $p$-set $P$, while $B_\alpha^{p,p}$, for a function $f \in AC([a,b])$, is called the operation $B$ ($B$-op) of $f$ of order $\alpha$ and $p$-set $P$.
A theory of the fractional calculus of variations started in 1996 with works of Riewe [10], in order to better describe nonconservative systems in mechanics. After that several approaches have been developed to generalize the least action principle and the Euler–Lagrange equations to include fractional derivatives [2, 6, 7, 9]. Here we study incommensurate fractional variational problems with generalized fractional operators.

Let us consider the problem of finding a function $y = t \mapsto y(t)$, $t \in [a, b]$, that gives an extremum (minimum or maximum) to the functional $\mathcal{J}(y)$ defined by

$$K_p^\alpha \left[ t \mapsto F \left( t, y(t), y'(t), B^\alpha_{P_1} [y](t), \ldots, B^\alpha_{P_n} [y](t), K_{R_1}^\gamma [y](t), \ldots, K_{R_n}^\gamma [y](t) \right) \right] (b)$$

when subject to the boundary conditions

$$y(a) = y_a, \quad y(b) = y_b,$$

where $\alpha, \beta_i, \gamma_i \in (0, 1)$, $P_1 = < a, b, 1, 0 >$, $P_i = < a, t, b, p_i, q_i >$, $R_i = < a, t, b, r_i, s_i >$, $i = 1, \ldots, n$. For simplicity of notation we introduce the operator \{ $P_{\beta,\gamma}$ defined by

$$\{ y \}^{\beta,\gamma}_{P_{\beta,\gamma}} (t) = (t, y(t), y'(t), B^\beta_{P_1} [\tau \mapsto y(\tau)] (t), K_{R_1}^\gamma [\tau \mapsto y(\tau)] (t)), \quad \text{where}$$

$$B^\beta_{P_1} = (B^\beta_{P_1}, \ldots, B^\beta_{P_n}), \quad K_{R_1}^\gamma = (K_{R_1}^\gamma, \ldots, K_{R_n}^\gamma).$$

The operator $K_{P_{\beta,\gamma}}$ has kernel $k_\alpha(x, t)$, and for $i = 1, \ldots, n$ operators $B^\beta_{P_i}$ and $K_{R_i}^\gamma$ have kernels $h_{1-\beta_i}(t, \tau)$ and $h_{\gamma_i}(t, \tau)$, respectively. In the sequel we assume that:

(H1) Lagrangian $F \in C^1 ([a, b] \times \mathbb{R}^{2n+2}; R)$;

(H2) for $i = 1, \ldots, n$ functions $A^\beta_{P_i}$ \[ t \mapsto k_\alpha(b, \tau) \partial_{\beta+i} F \{ y \}^{\beta,\gamma}_{P_{\beta,\gamma}} (\tau), \]

$K_{R_i}^{\gamma_i}$ \[ t \mapsto k_\alpha(b, \tau) \partial_{\gamma+i} F \{ y \}^{\beta,\gamma}_{P_{\beta,\gamma}} (\tau), \]

$D \left[ t \mapsto \partial_3 F \{ y \}^{\beta,\gamma}_{P_{\beta,\gamma}} (t)k_\alpha(b, t) \right]$ and $t \mapsto k_\alpha(b, t) \partial_2 F \{ y \}^{\beta,\gamma}_{P_{\beta,\gamma}} (t)$ are continuous on $(a, b)$;

(H3) functions $t \mapsto \partial_3 F \{ y \}^{\beta,\gamma}_{P_{\beta,\gamma}} (t)k_\alpha(b, t)$, $K_{P_i}^{1-\beta_i} \left[ t \mapsto k_\alpha(b, \tau) \partial_{\beta+i} F \{ y \}^{\beta,\gamma}_{P_{\beta,\gamma}} (\tau) \right] \in AC([a, b]), i = 1, \ldots, n;

(H4) kernels $k_\alpha(x, t)$, $h_{1-\beta_i}(t, \tau)$ and $h_{\gamma_i}(t, \tau)$, $i = 1, \ldots, n$, are such that we are in conditions to use Theorems 9, 10 and 11 of [9].

**Theorem 0.4.** If $y$ is a solution to problem (1)–(2), then $y$ satisfies the generalized Euler–Lagrange equation

$$k_\alpha(b, t) \partial_2 F \{ y \}^{\beta,\gamma}_{P_{\beta,\gamma}} (t) - \frac{d}{dt} \left( \partial_3 F \{ y \}^{\beta,\gamma}_{P_{\beta,\gamma}} (t)k_\alpha(b, t) \right)$$

$$- \sum_{i=1}^n A^\beta_{P_i} \left[ t \mapsto k_\alpha(b, \tau) \partial_{\beta+i} F \{ y \}^{\beta,\gamma}_{P_{\beta,\gamma}} (\tau) \right] (t)$$

$$+ \sum_{i=1}^n K_{R_i}^{\gamma_i} \left[ t \mapsto k_\alpha(b, \tau) \partial_{\gamma+i} F \{ y \}^{\beta,\gamma}_{P_{\beta,\gamma}} (\tau) \right] (t) = 0 \quad (3)$$

for all $t \in (a, b)$.

Assume now that in problem (1)–(2) the boundary conditions (2) are substituted by

$$y(a) \text{ is free} \quad \text{and} \quad y(b) = y_b.$$
Theorem 0.5. If \( y \) is a solution to the problem of extremizing functional (1) with (4) as boundary conditions, then \( y \) satisfies the Euler–Lagrange equation (3). Moreover, the extra natural boundary condition
\[
\partial_3 F \{ y \}_{P_D,R_I}^{\beta,\gamma} (a) + \sum_{i=1}^{n} K_{P_r}^{1-\beta_i} \left[ \tau \mapsto \partial_{i+3} F \{ y \}_{P_D,R_I}^{\beta,\gamma} (\tau) k_\alpha (b, \tau) \right] (a) = 0 \tag{5}
\]
holds.

Let us consider now the isoperimetric problem: among all functions \( y : [a, b] \to \mathbb{R} \) satisfying boundary conditions (2) and an isoperimetric constraint of the form
\[
\mathcal{I} (y) = K_{P_r}^{\beta} \left[ G \{ y \}_{P_D,R_I}^{\beta,\gamma} \right] (b) = \xi, \tag{6}
\]
where \( \xi \in \mathbb{R} \) is fixed, we look for the one that extremizes (i.e., minimizes or maximizes) functional (1). We assume that functional (6) satisfies hypotheses (H1)–(H4).

Definition 0.6. An admissible function \( y \in C^1 ([a, b], \mathbb{R}) \) is said to be an extremal for \( \mathcal{I} \) if it satisfies the Euler–Lagrange equation (3) associated with functional in (6).

Theorem 0.7. If \( y \) is a solution to the isoperimetric problem defined by (1)–(2), (6), and is not an extremal for \( \mathcal{I} \), then there exists a real constant \( \lambda \) such that
\[
k_\alpha (b, t) \partial_2 H \{ y \}_{P_D,R_I}^{\beta,\gamma} (t) - \frac{d}{dt} \left( \partial_3 H \{ y \}_{P_D,R_I}^{\beta,\gamma} (t) k_\alpha (b, t) \right) - \sum_{i=1}^{n} A_{P_r}^{\beta_i} \left[ \tau \mapsto k_\alpha (b, \tau) \partial_{i+3} H \{ y \}_{P_D,R_I}^{\beta,\gamma} (\tau) \right] (t)
+ \sum_{i=1}^{n} K_{P_r}^{\gamma_i} \left[ \tau \mapsto k_\alpha (b, \tau) \partial_5 H \{ y \}_{P_D,R_I}^{\beta,\gamma} (\tau) \right] (t) = 0 \tag{7}
\]
for all \( t \in (a, b) \), where \( H (t, y, u, v, w) = F (t, y, u, v, w) - \lambda G (t, y, u, v, w) \) and \( P_r^* = \langle a, t, b, q_j, p_j \rangle \).

As an application, we extend some of the recent results of [3, 4], where the fractional action-like variational approach (FALVA) was proposed to model dynamical systems. FALVA functionals are particular cases of (1), when it does not depend on \( B \)-op and \( K \)-op. The Caldirola–Canai Lagrangians, that describe dynamical oscillatory systems with exponentially increasing time dependent mass are considered: for details see [9].

REFERENCES


