AN IMPROVED MAX-PLUS EIGENVECTOR METHOD FOR THE APPROXIMATION OF NONLINEAR $\mathcal{L}_2$-GAIN BOUNDS*

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EXTENDED ABSTRACT. This paper considers computation of tight transient and gain bound estimates associated with a nonlinear $\mathcal{L}_2$-gain property via a max-plus eigenvector method. In particular, a new auxiliary optimal control problem is developed to overcome a specific continuity issue that arises in the application of max-plus eigenvector methods to a class of problems identified in [6].

1. Nonlinear $\mathcal{L}_2$-gain and associated tight gains. Attention is restricted to nonlinear dynamical systems of the form

$$\Sigma : \left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x(t)) + g(x(t))w(t), \quad x(0) = x, \\ z(t) & = & h(x(t)). \end{array} \right. \tag{1.1}$$

Recalling [2], [5], [6], system (1.1) has nonlinear $\mathcal{L}_2$-gain with transient / gain bound pair $(\beta, \gamma) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$ if

$$\|z\|^2_{\mathcal{L}_2[0,T]} \le \beta(|x|) + \gamma\left(\|w\|^2_{\mathcal{L}_2[0,T]}\right) \tag{1.2}$$

for all $x \in \mathbb{R}^n$, $w \in \mathcal{L}_2[0,T]$, $T \ge 0$, where a function $\gamma : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is of class $\bar{\mathcal{K}}$ if it is non-decreasing, right continuous at 0 and satisfies $\gamma(0) = 0$. Let $\Pi^\Sigma \subset \bar{\mathcal{K}} \times \bar{\mathcal{K}}$ denote the set of all transient / gain bound pairs (β, γ) for which system (1.1) satisfies the nonlinear $\mathcal{L}_2$-gain property (1.2), and denote

$$\mathcal{B}^\Sigma(\gamma) \doteq \left\{ \beta \in \bar{\mathcal{K}} \middle| \beta, \gamma) \in \Pi^\Sigma \right\}, \; \mathcal{G}^\Sigma(\beta) \doteq \left\{ \gamma \in \bar{\mathcal{K}} \middle| \beta, \gamma) \in \Pi^\Sigma \right\}.$$

Three specific quantities are of particular interest [8] for the study of the nonlinear $\mathcal{L}_2$-gain property (1.2), namely

$$\begin{array}{rcl} \gamma_*(\xi) & \doteq & \sup\limits_{T \ge 0} \sup\limits_{\|w\|^2_{\mathcal{L}_2[0,T]} = \xi} \left\{ \|z\|^2_{\mathcal{L}_2[0,T]} \middle| \;\; (1.1) \text{ holds}, \; x(0) = 0 \right\}, \\ \gamma_*^\beta(\xi) & \doteq & \inf \left\{ \gamma(\xi) \middle| \gamma \in \mathcal{G}^\Sigma(\beta) \right\}, \\ \beta_*^\gamma(s) & \doteq & \inf \left\{ \beta(s) \middle| \beta \in \mathcal{B}^\Sigma(\gamma) \right\}. \end{array}$$

It is shown [2], [8] the computation of various tight nonlinear $\mathcal{L}_2$ transient / gain bounds associated with system (1.1) is reduced to solving an optimal control problem

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with value function $W : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, where

$$W(x,\xi) \doteq \sup_{T\geq 0} \sup_{||w||^2_{\mathcal{L}_2[0,T]}=\xi} \left\{ ||z||^2_{\mathcal{L}_2[0,T]} \,\middle|\, \text{(1.1) holds with } x(0)=x \right\}. \tag{1.3}$$

In particular, the tight gain bounds $\gamma_*, \gamma_*^\beta, \beta_*^\gamma$ are expressed via (1.3) by

$$\begin{aligned} \gamma_*(s) &= W(0,s), \\ \gamma_*^\beta(s) &= \sup_{x\in\mathbb{R}^n} \{W(x,s) - \beta(|x|)\}, \\ \beta_*^\gamma(s) &= \sup_{|x|\leq s} \sup_{\xi\geq 0} \{W(x,\xi) - \gamma(\xi)\}. \end{aligned}$$

2. A max-plus eigenvector method implementation issue. The max-plus eigenvector method developed in [4] exploits max-plus linearity of the dynamic programming evolution operator. In [6], preliminary efforts in the application of this approach to the computation of (1.3) are presented. The first issue encountered there is that dynamic programming is not directly applicable to (1.3) due to the presence of the input norm constraint. Various techniques [2], [3], [8] have been applied to lift this input norm constraint and transform (1.3) into an equivalent unconstrained optimal control problem. However, difficulties still remain in applying a max-plus eigenvector method to this unconstrained optimal control problem. Here, a particular reformulation of the optimal control problem (1.3) is examined with a view to explaining how these difficulties arise. An auxiliary problem is then formulated so that this issue can be resolved. To this end, define new augmented dynamics

$$\Sigma^c : \left\{ \begin{array}{rcl} \left[\begin{array}{c} \dot{x}(t) \\ \dot{\xi}(t) \end{array}\right] &=& \left[\begin{array}{c} f(x(t)) + g(x(t))w(t) \\ -|w(t)|^2 \end{array}\right], \\ y(t) &=& \sigma(\xi(t))h(x(t)) \end{array} \right. \tag{2.1}$$

with

$$\sigma(\xi) = \left\{ \begin{array}{ll} 1, & \xi \geq 0, \\ 0, & \xi < 0. \end{array} \right.$$

A new optimal control problem is defined with value function $\overline{W} : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, where

$$\overline{W}(x,\xi) \doteq \sup_{T\geq 0} \sup_{w\in\mathcal{L}_2[0,T]} \left\{ ||y||^2_{\mathcal{L}_2[0,T]} \,\middle|\, \begin{array}{c} \text{(2.1) hold with} \\ x(0)=x,\ \xi(0)=\xi \end{array} \right\}. \tag{2.2}$$

It can be shown that $\overline{W}$ of (2.2) is equivalent to W of (1.3), with $W(x,\xi) = \overline{W}(x,\xi)$ for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$. Appling dynamic programming [1], $\overline{W}$ is a fixed point of an evolution operator $\mathcal{S}_\tau$, where $\mathcal{S}_\tau$ is defined by

$$\mathcal{S}_\tau[\phi](x,\xi) \doteq \sup_{w\in\mathcal{L}_2[0,\tau]} \left\{ ||y||^2_{\mathcal{L}_2[0,\tau]} + \phi(x(\tau),\xi(\tau)) \right\}.$$

That is, $\overline{W} = \mathcal{S}_\tau[\overline{W}]$. The max-plus eigenvector method [4], [6], [7] approximates this variational fixed point relation via a finite dimensional max-plus eigenvector equation given by

$$0 \otimes a = B \otimes a, \tag{2.3}$$

in which, $a_i \doteq \mathcal{D}_{\psi_i}\overline{W}$, $B_{ij} \doteq \mathcal{D}_{\psi_i}\mathcal{S}_\tau[\psi_j]$, $\mathcal{D}_\psi\phi \doteq -\sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}_{\geq 0}}\{\psi(x,\xi)-\phi(x,\xi)\}$, and $\psi_j \in \mathcal{B}$ denotes the j^{th} (countable) basis function

$$\psi_j(x,\xi) \doteq -\frac{1}{2}(x-x_j)'C_j(x-x_j) - \frac{q_j}{2}|\xi-\xi_j|^2, \tag{2.4}$$

where $C_j \in \mathbb{R}^{n\times n}$, $C_j = C_j' > 0$ and $q_j > 0$. This eigenvector equation (2.3) can be solved by a simple power method, yielding $a = B^{\otimes N} \otimes 0$, for some positive integer $N \in \mathbb{N}$. Let $\overline{W}_j : \mathbb{R}_{\geq 0}\times\mathbb{R}^n\times\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ denote the value function of the finite horizon optimal control problem defined by $\overline{W}_j(\tau,x,\xi) \doteq \mathcal{S}_\tau[\psi_j](x,\xi)$. It is well-known [1] that $\overline{W}_j$ satisfies the non-stationary Hamilton-Jacobi-Bellman partial differential equation (HJB PDE)

$$\frac{\partial\overline{W}_j}{\partial\tau}(\tau,x,\xi) = H(x,\xi,\nabla_x\overline{W}_j(\tau,x,\xi),\nabla_\xi\overline{W}_j(\tau,x,\xi)) \tag{2.5}$$

subject to initialization $\overline{W}_j(0,x,\xi) = \psi_j(x,\xi)$, where $H : \mathbb{R}^n\times\mathbb{R}_{\geq 0}\times\mathbb{R}^n\times\mathbb{R} \to \mathbb{R}$ is the Hamiltonian

$$\begin{aligned} H(x,\xi,p,q) &\doteq -|\sigma(\xi)h(x)|^2 - \langle p, f(x)\rangle - \pi(x,\xi,p,q), \\ \pi(x,\xi,p,q) &\doteq \begin{cases} \frac{|g(x)'\,p|^2}{4\,q}, & q>0, \\ 0, & q=0 \text{ and } g'(x)\,p = 0, \\ \infty, & q<0 \text{ or } (q=0 \text{ and } g'(x)\,p \neq 0). \end{cases} \end{aligned} \tag{2.6}$$

The existence of a C^2 solution of (2.5) is in general very useful in the implementation of max-plus eigenvector methods [4]. Where this is the case, $\overline{W}_j(\tau,x,\xi)$ can be approximated by the first order Taylor series expansion

$$\overline{W}_j(\tau,x,\xi) \approx \psi_j(x,\xi) + \tau\, H(x,\xi,\nabla_x\psi_j(x,\xi),\nabla_\xi\psi_j(x,\xi)). \tag{2.7}$$

However, it may be shown that such a C^2 solution for (2.5) cannot exist, as is the case in [6]. In particular, setting $\tau = 0$ in (2.5) yields

$$\frac{\partial\overline{W}_j}{\partial\tau}(0,x,\xi) = H(x,\xi,\nabla_x\psi_j(x,\xi),\nabla_\xi\psi_j(x,\xi)),$$

while (2.4) implies that $\nabla_x\psi_j(x,\xi) = -C_j(x-x_j)$ and $\nabla_\xi\psi_j(x,\xi) = -q_j(\xi-\xi_j)$. So applying (2.6), the Hamiltonian $H(x,\xi,\nabla_x\psi_j(x,\xi),\nabla_\xi\psi_j(x,\xi))$ is unbounded on

$$\mathcal{O} \doteq \left\{ (x,\xi)\in\mathbb{R}^n\times\mathbb{R}_{\geq 0} \;\middle|\; \begin{array}{c} q_j(\xi-\xi_j) > 0 \text{ or} \\ q_j(\xi-\xi_j) = 0 \text{ and } g'(x)C_j(x-x_j) \neq 0 \end{array} \right\},$$

implying that $\overline{W}_j(\tau,x,\xi)$ is not differentiable with respect to τ at $\tau = 0$, $(x,\xi)\in\mathcal{O}$.

3. A new auxiliary control problem and max-plus eigenvector method.

It can be seen that the continuity issue described in Section 2 arises due to the absence of a direct penalty on the input in the cost of (2.2). This issue can be alleviated by modifying the cost function of (2.2) to include such a direct penalty. To this end, consider a new optimal control problem with value function $\widetilde{W}_{\lambda,\delta} : \mathbb{R}^n\times\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by

$$\widetilde{W}_{\lambda,\delta}(x,\xi) \doteq \sup_{T\geq 0}\sup_{w\in\mathcal{L}_2[0,T]} \left\{ \begin{array}{c} \lambda\|y\|^2_{\mathcal{L}_2[0,T]} \\ -\delta\|w\|^2_{\mathcal{L}_2[0,T]} \end{array} \;\middle|\; \begin{array}{c} \text{(2.1) holds with} \\ X(0)=x, \\ \xi(0)=\xi \end{array} \right\}, \tag{3.1}$$

where $\lambda, \delta > 0$ are free parameters. By assuming that a specific the *zeros-state-input sensitivity* (0-SIS) condition holds with sensitivity pair $(\varepsilon_0, \sigma_0) \in \mathbb{R}^2_{>0}$ (see [7], [8]), and setting parameters λ, δ such that $\lambda\varepsilon_0 > \delta$, the value function $\widetilde{W}_{\lambda,\delta}$ may be related [7] to the value function W in (1.3) by

$$W(x,\xi) = \frac{\widetilde{W}_{\lambda,\delta}(x,\xi) + \delta\xi}{\lambda}, \ \forall\ (x,\xi) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}. \tag{3.2}$$

Applying dynamic programming [1], $\widetilde{W}_{\lambda,\delta}$ is the fixed point of another evolution operator $\widetilde{\mathcal{S}}_\tau$, where

$$\widetilde{\mathcal{S}}_\tau[\phi](x,\xi) \doteq \sup_{w\in\mathcal{L}_2[0,\tau]} \left\{ \lambda||y||^2_{\mathcal{L}_2[0,\tau]} - \delta||w||^2_{\mathcal{L}_2[0,\tau]} + \phi(x(\tau),\xi(\tau)) \right\}.$$

That is, $\widetilde{W}_{\lambda,\delta} = \widetilde{\mathcal{S}}_\tau[\widetilde{W}_{\lambda,\delta}]$. The value function for the finite horizon optimal control problem $\widetilde{W}_j(\tau,x,\xi) \doteq \widetilde{\mathcal{S}_\tau}[\psi_j](x,\xi)$ satisfies the HJB PDE of the form (2.5), where the new Hamiltonian is

$$\begin{aligned} \widetilde{H}(x,\xi,p,q) &\doteq -\lambda|\sigma(\xi)h(x)|^2 - \langle p, f(x)\rangle - \widetilde{\pi}(x,\xi,p,q), \\ \widetilde{\pi}(x,\xi,p,q) &\doteq \begin{cases} \frac{|g(x)'\,p|^2}{4\,(q+\delta)}, & q+\delta > 0, \\ 0, & q+\delta = 0 \text{ and } g'(x)\,p = 0, \\ \infty, & q+\delta < 0 \text{ or } (q+\delta = 0 \text{ and } g'(x)\,p \neq 0). \end{cases} \end{aligned} \tag{3.3}$$

For the Hamiltonian (3.3), $\widetilde{H}(x,\xi,\nabla_x\psi_j(x,\xi),\nabla_\xi\psi_j(x,\xi))$ is unbounded on

$$\widetilde{\mathcal{O}} \doteq \left\{ (x,\xi) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0} \;\middle|\; \begin{array}{c} q_j(\xi - \xi_j) > \delta \text{ or} \\ q_j(\xi-\xi_j) = \delta \text{ and } g(x)'C_j(x-x_j) \neq 0 \end{array} \right\}.$$

However, the free parameters λ and δ can be chosen such that $q_j(\xi - \xi_j) < \delta$, yielding a Hamiltonian that is continuous on the computational domain of interest, denoted by $\mathbb{R}^n \times [0, \bar{\xi}]$. That is, $\emptyset = \widetilde{\mathcal{O}} \cap (\mathbb{R}^n \times [0,\bar{\xi}])$.

THEOREM 3.1. *Given $\bar{\xi}, \delta \in \mathbb{R}_{>0}$ satisfying $q_j\bar{\xi} < \delta$, the Hamiltonian $\widetilde{H}(x,\xi,\nabla_x\widetilde{W}_j,\nabla_\xi\widetilde{W}_j)$ is bounded on $\mathbb{R}^n \times [0,\bar{\xi}]$.*

In summary, $\widetilde{W}_{\lambda,\delta}$ of (3.1) may be computed using a max-plus method (2.3), while W of (1.3) may be recovered from the computed $\widetilde{W}_{\lambda,\delta}$ of (3.2).

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