Geometric and algebraic conditions for j-controllable behaviors.

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For most kinds of multi-dimensional input-state-output systems, evolution of the state is given by a single direction or a collection of independent directions. The equations indicate that the state is "freely driven by the input." In particular, one can uncouple the input from both the state and output in the form of linear pencils. A crucial component of this uncoupling is that the free components may be arbitrarily chosen at all points in time, i.e. time or shift invariance. In this work, we consider the realm of trajectories that, while not completely free, permit some freedom along particularly nice subsets of the time domain. This generalization indicates that shift-invariance over $\mathbb{N}^d$ is just a special case.

The cornerstone of our derivations are Livšic systems, which are input-state-output systems that satisfy the following independent evolution equations along *each* direction

$$\begin{aligned} x(t_1+1,t_2) &= A_1x(t_1,t_2) + B_1u(t_1,t_2) \\ x(t_1,t_2+1) &= A_2x(t_1,t_2) + B_2u(t_1,t_2) \\ y(t_1,t_2) &= Cx(t_1,t_2) + Du(t_1,t_2) \end{aligned}$$

where x, u, and y are respectively the state, input and output trajectories. Consistency is enforced by compatibility conditions and linkage conditions placed on the the matrices and trajectories to yield what are known as the **vessel conditions**. One may consult [1] and [2] for a brief overview of vessels in, respectively, the discrete and continuous time setting.

We begin with the observation is that there are controllable Livšic systems. Furthermore, identification of this property can be performed with the standard arsenal of tests, such as a Hautus test, that one enjoys in the classical (free-input) setting. While such a property is contrary to conventional behavioral system theory (autonomous systems are *not* controllable), a further analysis of the property, in the setting of Livšic systems, reveals an intrinsic property of controllable systems that holds generally in the behavioral setting.

Building upon the exemplary theory developed by Oberst in [4] to its full extent (i.e. for behaviors over affine domains), we present d-dimensional versions of free variables, $\mathcal{D}_r$-controllability (image representation over $\mathcal{D}_r$), j-controllability and transfer classes for autonomous behaviors. We demonstrate that this generalized theory is completely consistent with the classical theory, i.e., the classical theory is a special case (the autonomy degree zero case) of behaviors over $\mathbb{C}[z_1,\dots,z_d]$ rather than $\mathcal{D}_r$. One may consult [3] for a complete treatment of the following definitions and results.

While our goal is to discuss Livšic systems and their role as a state space representation of two-dimensional behaviors, a crucial component is the machinery necessary for making this connection. In particular, working with behaviors considered over an algebraic variety with operators in its associated coordinate ring. We define this entity as follows.

Definition 1. Let $\mathcal{B} = \ker_{\mathcal{A}}(R)$ be an autonomous behavior and $I \subset \mathrm{Ann}(\mathcal{B})$ be a prime ideal. Define the **reduced ring** $\mathcal{D}_r = \mathcal{D}/I$ and let $\phi : \mathcal{D} \to \mathcal{D}_r$ be the canonical quotient map. The **reduced signal space**

associated to $\mathcal{D}_r$ is $\mathcal{A}_r = \{w \in \mathcal{A} : pr = 0 \text{ for all } p \in I\}$. We define the behavior **reduced** by I as the $\mathcal{D}_r$-module given by $\mathcal{B}_r = \{w \in \mathcal{A}_r : \phi(R)w = 0\}$.

As in the classical setting, reduced behaviors benefit from many of the same properties; in particular, we have the following equivalences.

Theorem 1. *Let $\mathcal{B}_r = \ker_{\mathcal{A}_r}(R)$ be a reduced behavior, where the associated ring $\mathcal{D}_r$ is an affine domain, and $\mathcal{M}_r = Coker_{\mathcal{D}_r}(R^T)$ be its dual module. The following are equivalent.*

1. *$\mathcal{M}_r$ is torsion-free*
2. *$\mathcal{B}_r$ has an image representation.*
3. *R is generalized factor left prime over $\mathcal{D}_r$*
4. *$\mathcal{B}_r$ is minimal in its transfer class.*
5. *$\mathcal{B}_r$ is divisible.*

Following precedent, we call a behavior $\mathcal{D}_r$-controllable if it satisfies any of the equivalent properties in Theorem 1.

This is the algebraic version of controllability. As in the classical setting, there is a time-domain version that brings to light the intuitive properties of controllability. For brevity, we define $\partial\text{Ann}(\mathcal{B})$ as the initial condition set in the time domain for the canonical Cauchy problem. For the other definitions we refer the reader to [3] for further treatment.

Extending the work of Oberst [4], Willems [6], Zerz and Wood [7] and Rocha [5] yields the following definition and results.

Definition 2. For $j \in \{1, \ldots, d\}$ and $\mathcal{B}$ a d-dimensional behavior we say that $\mathcal{B}$ is **j-controllable** if there exists a j-dimensional lattice $\mathcal{L}[u]$, $u \subset \{z_1, \ldots, z_d\}$ and $|u| = j$, (the **motion sublattice**) such that $\mathcal{L}[u] \subset \partial\text{Ann}(\mathcal{B})$ and, for any $T_1 \subset \partial\text{Ann}(\mathcal{B})$ such that $\dim(T_1) < j$ and *finite* subset $J \subset \mathbb{N}^d$, there exists an integer (the **separation distance**) $\tau(T_1 + J) \geq 0$ such that, for any $w_1, w_2 \in \mathcal{B}$ and $b \in \mathcal{L}[u]$ with $d(T_1 + J, b + \mathbb{N}^d) > \tau(T_1 + J)$, there exists $w \in \mathcal{B}$ such that

$$w(t) = \begin{cases} w_1(t) & t \in T_1 + J \\ w_2(t - b) & t \in b + \mathbb{N}^d. \end{cases}$$

Although not immediately obvious, Definition 2, when restricted to the case when $\text{Ann}(\mathcal{B}) = 0$, is the same definition of controllability as presented by Zerz and Wood [7]. Furthermore, the following theorems closely parallel the classical (free-input) setting.

Theorem 2. *Let $\mathcal{B} = \ker_{\mathcal{A}}(R)$ be a given behavior. Provided $\dim(\mathcal{B}) = j$, $\mathcal{B}$ is j-controllable, and Ann($\mathcal{B}$) is a prime ideal, then $\mathcal{B}_r$ is $\mathcal{D}_r$-controllable.*

Theorem 3. *Let $\mathcal{B} \subset \mathcal{A}^q$ be a behavior. If Ann($\mathcal{B}$) is a prime ideal, the reduced signal space $\mathcal{A}_r$ is j-controllable with motion sublattice $\mathcal{L}[u]$, $u \subset \{z_1, \ldots, z_d\}$, $|u| = j$, $\mathcal{L}[u] \subset \partial Ann(\mathcal{B})$, and $\mathcal{B}_r$ has image representation M over its reduced ring, then $\mathcal{B}$ is j-controllable with motion sublattice $\mathcal{L}[u]$.*

A problem that arises is determining whether or not the reduced signal space $\mathcal{A}_r$ is j-controllable with motion sublattice $\mathcal{L}[u]$. Constructing a test for this property is not immediately obvious; especially since both the definition and property depend upon the polynomial ordering used in the Gröbner basis.

While not an exhaustive answer to the question, we provide conditions for j-controllability in the special case when $\text{Ann}(\mathcal{B})$ is both a principal and prime ideal. Furthermore, this test is independent of polynomial ordering. This will set the stage for addressing how j-controllability intimately links the geometric, algebraic and trajectory-based versions of controllability.

References

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