ON THE "WEAK" PROPER SOLUTIONS OF TANGENTIAL INTERPOLATION PROBLEMS

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EXTENDED ABSTRACT. Fuhrmann in [2] introduced the notion of the so-called weak solutions tangential interpolation problem. Accordingly, the rational function $T(z)$ is a "weak" solution of a left tangential interpolation problem if its polynomial coprime factorization $T(z) = P(z)Q(z)^{-1}$ satisfies the equations $\eta_i Q(\lambda_i) = \xi_i P(\lambda_i)$ for a given set of interpolation nodes $\lambda_1, \dots, \lambda_k$, interpolation directions $\xi_1, \dots, \xi_k$ and interpolation values $\eta_1, \dots, \eta_k$. Obviously, if T is analytic at the interpolation nodes it can be written in the usual form $\xi_i T(\lambda_i) = \eta_i$, $i = 1, \dots, k$.

We generalize the notion of "weak" solutions to higher order interpolation and describe the minimal degree proper solutions of this generalized interpolation problem.

Tangential interpolation problem for proper rational functions. We assume the elements of all matrices belong to the same real or complex field.

Let $\mathcal{A}, U, V$ be given, where $\mathcal{A}$ is a square matrix. Then the proper rational function T – analytic on the spectrum of $\mathcal{A}$ – satisfies the "usual" left interpolation problem

$$\frac{1}{2\pi i}\int_\Gamma (zI - \mathcal{A})^{-1} U T(z) dz = V \tag{0.1}$$

for any closed curve Γ containing $\sigma(\mathcal{A}_1)$ but not the poles of T, if and only if there exist a matrix Y such that for a minimal realization $T = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ the following equation holds:

$$[Y, U]\begin{bmatrix} A & B \\ C & D \end{bmatrix} = [\mathcal{A}Y, V] \tag{0.2}$$

From the equation (0.2) it follows immediately that

$$\begin{aligned} Y\left[B, AB, ...A^k B\right] &= \left[V, \mathcal{A}V, ..., \mathcal{A}^k V\right] \\ &\quad - \left[U, \mathcal{A}U, ..., \mathcal{A}^k U\right]\begin{bmatrix} D & CB & CAB & \dots & CA^{k-1}B \\ 0 & D & CB & \dots & CA^{k-2}B \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & D & CB \\ 0 & \dots & \dots & 0 & D \end{bmatrix} \end{aligned} \tag{0.3}$$

for any $k \geq 1$.

Based on this observation the following theorem can be proved:

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THEOREM 0.1. *Let $\mathcal{A}, U, V$ be given where the matrices U, V are of dimension $n \times p$ and $n \times m$ and let $\alpha(z) = \sum_{i=0}^{k} \alpha_i z^i$ and $\beta(z) = \sum_{i=0}^{k} \beta_i z^i$ be right-coprime polynomial matrices with α column reduced assuming that the degree of each column of β is not greater than that of the corresponding column of α. Then the function $T(z) = \beta(z)\alpha(z)^{-1}$ – assuming it is analytic on $\sigma(\mathcal{A})$ – interpolates $\mathcal{A}, U, V$ if and only if*

$$\sum_{i=0}^{k} \mathcal{A}^i V \alpha_i = \sum_{i=0}^{k} \mathcal{A}^i U \beta_i \tag{0.4}$$

Concerning minimal degree proper solutions the following bound can be derived immediately.

PROPOSITION 0.2. *Let $(\mathcal{A}, U, V)$ be given with $(\mathcal{A}, U)$ observable, and let $\kappa_1, ..., \kappa_m$ be the controllability indexes of V in the pair $(\mathcal{A}, [U, V])$. Then the McMillan-degree of any rational* **proper** *solution Q of the interpolation Problem 0.1 cannot be smaller than $\sum_{i=1}^{m} \kappa_i$.*

We show that analysing the "positions" of the left eigenvectors of the matrix $\mathcal{A}$ the equality in the previous lower bound can be characterized, as well.

Note that characterizing the minimal degree not necessarily proper solutions of the interpolation problem all the controllability indices of $(\mathcal{A}, [U, V])$ should be taken into account. See Antoulas et al. [1].

Introducing the notion of the weak solutions of tangential interpolation problems the general case when the function Q is not necessarily analytic on $\sigma(\mathcal{A})$ can be handled.

Namely, let λ be a fixed complex number and consider a function $T(z)$ of size $p \times m$ possibly having a pole of order r at λ.

Assume that $\xi(z) = \sum_{j>-k} \xi_j (z-\lambda)^j$ and $\eta(z) = \sum_{j>-k} \eta_j (z-\lambda)^j$ are two Laurent-series around λ. If

$$\xi(z)T(z) - \eta(z) \quad \text{isanalyticat} \quad \lambda \tag{0.5}$$

then we say that the function $T(z)$ is a "weak" solution of the left tangential interpolation problem of order k taken at the interpolation node λ determined by the interpolation directions $\xi_{-k+1}, \xi_{-k+2}, \dots, \xi_0$ and interpolation directions $\eta_{-k+1}, \eta_{-k+2}, \dots, \eta_0$.

Let us define the following matrices

$$\mathcal{A} := \begin{bmatrix} \lambda & & & \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{bmatrix} \tag{0.6}$$

$$U := [\xi_{-k+1}, \xi_{-k+2}, \dots, \xi_0] \qquad V := [\eta_{-k+1}, \eta_{-k+2}, \dots, \eta_0] \tag{0.7}$$

where $\mathcal{A}$ is of size $k \times k$, the notion of "weak" solutions can be described as there exists a polynomial vector $\beta(z)$ such that

$$\left((zI - \mathcal{A})^{-1} U + \beta(z)\right) T(z) - (zI - \mathcal{A})^{-1} V \tag{0.8}$$

is analytic at λ.

This notion is in full analogy to the notion of zero directions when the function has a pole at the location of zero, as well.

It can be proved that for $k = 0$, i.e. for first order tangential interpolation this definition gives back the notion of weak solution of interpolation problem treated e.g. in Fuhrmann [2].

Straightforward calculation gives that condition (0.8) is equivalent to the existence of a matrix Y solving equation (0.2) providing the possibility of repeating the arguments considered in the case when the analyticity of $T(z)$ on $\sigma(\mathcal{A})$ was assumed a priori.

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