Lattice Theory in Phase Unwrapping and Timing Recovery

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1 Phase Unwrapping and Timing Recovery

We examine two classical problems in statistical signal processing—phase unwrapping and timing recovery—and their relationship to lattice theory. We present an overview of the authors' work in this area in the last few years, together with some new results regarding asymptotic properties of parameter estimates.

In certain contexts in speech, radar, imaging and communications signal processing, it is the instantaneous phase of a complex-valued signal that is of interest [1, 2, 3]. Consider a scenario in which a signal is observed in additive noise. The signal is decomposed into an instantaneous amplitude and phase, so that the measurement $x(t)$ can be expressed $x(t) = A(t)e^{i\phi(t)} + \xi(t)$ where $A(t) \geqslant 0$ is the instantaneous amplitude, $\phi(t)$ is the instantaneous phase and $\xi(t)$ is the noise. Of particular interest is the special case where:

1. the noisy signal is sampled only at a finite number, N, of time instants $t = nT_\mathrm{s}$, $n = 1, \ldots, N$, for some sampling period T_s, and
2. only the noisy instantaneous phase measurements $z_n = \angle x(nT_\mathrm{s})$ are available or useful, *e.g.*, it may be known *a priori* that $A(t) = 1$.

The instantaneous phase $\phi(t)$ can take any real value but its measurements are restricted to an interval of length 2π. The problem of *phase unwrapping* is to find integers k_n such that $z_n + 2\pi k_n$ closely approximates $\phi(nT_\mathrm{s})$. The problem is ill-posed unless there is some additional structure to $\phi(t)$. We examine the case where $\phi(t)$ is a polynomial in t. If the polynomial is 0th order, this is a problem of *direction*

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or *phase estimation*; 1st order, it is *frequency estimation*; 2nd and higher order, it is *chirp* or *polynomial-phase estimation.* We wish to estimate the coefficients of the polynomial. The k_n are nuisance parameters.

A related problem is that of *timing recovery*, which is of central importance to digital communications receivers but also in electronic warfare and computer analysis of music [4, 5, 6]. In this version of the problem, a number of occurrence times are observed. It is assumed that they are separated by multiples of a period T but they are not necessarily consecutive. Further, it is assumed that the measurements $\mathbf{t} = (t_1, \ldots, t_N)$ of occurrence times are noisy, so that the signal model is that $t_n = k_n T + \theta + \tau_n$ for $n = 1, \ldots, N$, where T is the period, θ the phase and the τ_n are measurement noise. Like the phase-unwrapping problem, the k_n can be considered nuisance parameters. Here, we will assume they are random variables such that $k_n = Y_n + k_{n-1}$ and the Y_n are independent and identically distributed with mean μ_Y.[1] Period and phase are the parameters of interest.

2 Lattices, the Lattice A_n^* and the NLP Problem

A lattice is a discrete additive subgroup of $\mathbb{R}^m$. A lattice Λ of rank n in $\mathbb{R}^m$, $m \geqslant n$, can be expressed as a set of points $\Lambda = \{\mathbf{Bk} \mid \mathbf{k} \in \mathbb{Z}^n\}$ where $\mathbf{B} \in \mathbb{R}^{n\times m}$, having full rank, is a *basis matrix.*

The lattice A_n^*, or rather family of lattices (indexed by n), is the dual of the root lattice A_n [7, §6.6]. It can be succinctly defined as the orthogonal projection of the $n+1$-dimensional integers into the zero-sum or zero-mean plane, *i.e.*, the plane on which coordinates sum to zero. A basis of A_n^* can be constructed from any n columns of the projection matrix $\mathbf{Q} = \mathbf{I} - \frac{1}{n+1}\mathbf{1}\mathbf{1}^T$ where $\mathbf{I}$ is the identity matrix and $\mathbf{1}$ is the $n+1$-dimensional column vector consisting of all 1s.

Given a point $\mathbf{y} \in \mathbb{R}^m$, the nearest lattice point (NLP) problem is to find the lattice point $\mathbf{x} \in \Lambda$ which is closest to it according to, for instance, the Euclidean norm. In general, the problem is *NP*-hard but, for the lattice A_n^*, it can be computed in $O(n)$ arithmetic operations [8].

3 Phase Unwrapping, Timing Recovery and the NLP

Phase unwrapping for noisy polynomial-phase signals and timing recovery from noisy occurrence times can be expressed as NLP problems on A_n^* and related lattices.

Consider the problem of least-squares (LS) direction estimation. In this case, the phase function is constant, say θ. It can be seen that the LS estimator of θ, conditioned on a certain choice of integers k_n, is the average of the values $z_n + 2\pi k_n$. The unconditional LS estimate is found by choosing those values for k_n which minimise the sample variance of $\{z_n + 2\pi k_n\}$. It can be shown that this is equivalent to finding the NLP in A_{N-1}^* to $\frac{1}{2\pi}\mathbf{Qz}$.

For frequency estimation, our phase function at $t = nT_\mathrm{s}$ has the form $\omega n + \theta$. It can be shown that the LS estimates of ω and θ are those found by choosing the values for k_n which minimise the residual for linear regression of $z_n + 2\pi k_n$ against n. This, again, can be formulated as an NLP problem, but in the lattice formed from the orthogonal projection of A_{N-1}^* along the vector $\mathbf{Qv}$ where $\mathbf{v} = (1, \ldots, N)^T$ [9]. Successively higher order polynomial-phase estimation can be posed as NLP problems on lattices that are successively projected from A_n^* [10].

[1] There are some other technical conditions on the distributions of τ_n and S_n that are required to prove Theorem 1. They are omitted here to preserve space

In timing recovery, the LS estimators of period and phase, conditioned on the k_n, are found from the linear regression of t_n against k_n. The unconditional LS estimates are found by choosing the k_n which minimise the residual of the regression. To ensure identifiability, it is necessary in this problem to restrict the range of the permissible period estimate to an octave, *i.e.*, an interval in which the upper bound is not more than twice the lower bound. Further, to ensure good asymptotic properties (as set out in Theorem 1), it is necessary to inversely weight the residual by the period. Computationally, this can be achieved by finding the closest lattice point in A_{N-1}^* to the line segment $f\mathbf{t}$, where f is allowed to vary over said octave [11].

4 Asymptotic Results

It is possible to conduct a rigorous statistical analysis of the LS estimators to obtain asymptotic results about their performance. Space allows for only one such result is presented here, relating to the asymptotic performance of timing recovery with the modified LS estimators of period, $\hat{T}_N$, and phase, $\hat{\theta}_N$.

Theorem 1. *The distribution of* $(N^{1/2}(\hat{\theta}_N - \theta), N^{3/2}(\hat{T}_N - T))^T$ *converges as* $N \to \infty$ *to the normal with mean* 0 *and covariance matrix*

$$\frac{12\sigma^2 T^2}{(1-h)^2 \mu_Y^2} \begin{pmatrix} \frac{1}{3}\mu_Y^2 & -\frac{1}{2}\mu_Y \\ -\frac{1}{2}\mu_Y & 1 \end{pmatrix},$$

where $h = f(1/2)$ *and* $f(x)$ *is the probability density function of the centred fractional part of* τ_n/T, i.e., *of* $\tau_n/T - \lfloor \tau_n/T \rceil$.

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