MDS 2D convolutional codes

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1 Introduction

In this paper we consider two-dimensional (2D) convolutional codes with finite support, i.e., convolutional codes which codewords have compact support indexed in $\mathbb{N}^2$ and take values in $\mathbb{F}^n$, where $\mathbb{F}$ is a finite field. Also, we denote by $\mathbb{F}[z_1, z_2]$ the ring of polynomials in 2 indeterminates with coefficients in $\mathbb{F}$.

Definition 1 ([3, 4]): A 2D finite support convolutional code $\mathcal{C}$ of rate k/n is a free $\mathbb{F}[z_1, z_2]$-submodule of $\mathbb{F}[z_1, z_2]^n$ with rank k. A full column rank matrix $G(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n \times k}$ whose columns constitute a basis for $\mathcal{C}$, i.e., such that

$$\begin{aligned} \mathcal{C} &= \mathrm{Im}_{\mathbb{F}[z_1,z_2]} G(z_1, z_2) \\ &= \left\{ \widehat{\boldsymbol{v}}(z_1, z_2) = G(z_1, z_2) \widehat{\boldsymbol{u}}(z_1, z_2) : \widehat{\boldsymbol{u}}(z_1, z_2) \in \mathbb{F}[z_1, z_2]^k \right\} \subseteq \mathbb{F}[z_1, z_2]^n, \end{aligned} \tag{1}$$

is called *an encoder* of $\mathcal{C}$. The elements of $\mathcal{C}$ are called codewords.

The (free) distance of a code defines its capability of error correction. We define the notion of distance for 2D convolutional codes as in [4]. The weight of $\widehat{\boldsymbol{v}}(z_1, z_2) = \sum_{(i,j) \in \mathbb{N}^2} \boldsymbol{v}(i, j) z_1^i z_2^j \in \mathbb{F}[z_1, z_2]^n$, with $\boldsymbol{v}(i, j) \in \mathbb{F}^n$ for $(i, j) \in \mathbb{N}^2$, is given by $\mathrm{wt}(\widehat{\boldsymbol{v}})) = \sum_{(i,j) \in \mathbb{N}^2} \mathrm{wt}(\boldsymbol{v}(i, j))$, where $\mathrm{wt}(\boldsymbol{v}(i, j))$ is the number of nonzero entries of $\boldsymbol{v}(i, j)$.

Definition 2: Given a 2D finite support convolutional code $\mathcal{C}$, the *distance* of $\mathcal{C}$ is defined as

$$\mathrm{dist}(\mathcal{C}) = \min \{ \mathrm{wt}(\widehat{\boldsymbol{v}}(z_1, z_2)) : \widehat{\boldsymbol{v}}(z_1, z_2) \in \mathcal{C} \setminus \{\boldsymbol{0}\} \}.$$

For a fixed rate k/n, maximum distance separable (MDS) block codes are the block codes with the maximum distance that a code with such rate can achieve. When considering convolutional codes, the maximum distance attained by a code with this rate is also influenced by another parameter: the degree of the code [2]. The degree (or complexity) of a one-dimensional (1D) convolutional code is defined as the maximal degree of the full size minors of any encoder of the code, or equivalently as the sum of the degrees of a column reduced encoder. However, when considering 2D convolutional codes these two notions are not equivalent. Let us consider the usual definition of (total) degree of a polynomial $\widehat{p}(z_1, z_2) = \sum_{(i,j)\in\mathbb{N}^2} p(i,j) z_1^i z_2^j$ as $\deg(\widehat{p}(z_1, z_2)) = \max\{i + j : p(i,j) \neq 0\}$. Note that encoders of the same convolutional code differ by unimodular matrices, and therefore their full size minors differ by a nonzero constant.

Definition 3 ([1]): Let $\mathcal{C}$ be a 2D convolutional code with an encoder $G(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n\times k}$. The internal degree of $G(z_1, z_2)$, denoted by $\delta_{\mathrm{i}}(G(z_1, z_2))$, is defined as the maximal degree of its full size minors and the *complexity* of $\mathcal{C}$, represented by $\widehat{\delta}_{\mathcal{C}}$, is defined as the internal degree of any encoder of $\mathcal{C}$.

Definition 4: Let $\mathcal{C}$ be a 2D convolutional code with an encoder $G(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n\times k}$ and ν_i the column degree of the ith column of $G(z_1, z_2)$, i.e, the maximum degree of the entries of the ith column. The *external degree* of $G(z_1, z_2)$, denoted by $\delta_{\mathrm{e}}(G(z_1, z_2))$, is defined as $\delta_{\mathrm{e}}(G(z_1, z_2)) = \sum_{i=1}^{k} \nu_i$ and the *degree* of $\mathcal{C}$, denoted by $\delta_{\mathcal{C}}$, is defined as the minimum of the external degrees of all the encoders of $\mathcal{C}$.

Note that if $G(z_1, z_2)$ is an encoder of $\mathcal{C}$, then $\delta_{\mathrm{i}}(G(z_1, z_2) \leq \delta_{\mathrm{e}}(G(z_1, z_2)$ and therefore $\widehat{\delta}_{\mathcal{C}} \leq \delta_{\mathcal{C}}$. As mentioned before, while for 1D convolutional codes, $\widehat{\delta}_{\mathcal{C}} = \delta_{\mathcal{C}}$, because they always admit column reduced encoders which external degree equals their internal degree, this is not the case for 2D convolutional codes as the following simple example shows. However, if $\mathcal{C}$ has an encoder with external degree equal to $\widehat{\delta}$, then we can immediately conclude that the degree of the code equals its complexity.

Example 1: The 2D convolutional code with encoder $G(z_1, z_2) = \begin{bmatrix} 1 & 0 \\ z_1 & z_2 \\ 1 & 1 \end{bmatrix}$, has complexity 1 but degree 2. ■

2 MDS 2D convolutional codes

Let $\mathcal{C}$ be a 2D convolutional code of rate k/n and degree δ. The following theorem gives an upper bound for the distance of such codes.

Theorem 1: *Let $\mathcal{C}$ be a 2D convolutional code of rate k/n and degree δ. Let $G(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n\times k}$ be an encoder of $\mathcal{C}$ with column degrees $\nu_1, \nu_2, \ldots, \nu_k$ such that $\nu_1 + \nu_2 + \cdots + \nu_k = \delta$. Then*

$$\operatorname{dist}(\mathcal{C}) \leq \frac{\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 2\right)}{2} n - \left(k - \delta + k \left\lfloor \frac{\delta}{k} \right\rfloor\right) + 1. \quad (2)$$

The upper bound of Theorem 1 is the generalization to 2D convolutional codes of the generalized Singleton bound for 1D convolutional codes [2]. We call this bound *2D generalized Singleton bound*. Moreover, we say that a 2D convolutional code of rate k/n and degree δ is called a *Maximum Distance Separable* (MDS) 2D convolutional code if its distance equals the upper bound (2). Next we give a construction of such codes. For that we need to consider the following definitions.

Definition 5: Let A be an $n \times \ell$ matrix over a finite field $\mathbb{F}$. We say that A is a *superregular* matrix if every square submatrix of A is nonsingular.

Definition 6: Let A be an $n \times \ell$ matrix over a finite field $\mathbb{F}$. A nontrivial minor of A corresponds to the determinant of a square submatrix of A which does not have zero elements. We say that A is a *nontrivial superregular* matrix if all its nontrivial minors are different from zero.

We will now consider a special class of superregular matrices that we will use to construct an $n \times k$ matrix over $\mathbb{F}[z_1, z_2]$ which will generate a 2D convolutional code with distance (2). Such matrices can be constructed for fields with sufficient number of elements.

Before to continue, we introduce the following notation, if $\mathcal{G} = \begin{bmatrix} \boldsymbol{g}_0 & \boldsymbol{g}_1 & \cdots & \boldsymbol{g}_{s-1} \end{bmatrix} \in \mathbb{F}^{r\times s}$ then

$$\bar{\mathcal{G}} = \begin{bmatrix} \boldsymbol{g}_0 \\ \boldsymbol{g}_1 \\ \vdots \\ \boldsymbol{g}_{s-1} \end{bmatrix} \in \mathbb{F}^{rs}$$

that is, $\bar{\mathcal{G}}$ is the column matrix obtained when we write the columns of matrix $\mathcal{G}$ one below the other.

Now, let n, k and δ be nonnegative integers and define

$$t = k - \delta + k\left\lfloor \frac{\delta}{k} \right\rfloor, \quad \ell_1 = \frac{\left(\lfloor \frac{\delta}{k} \rfloor + 1\right)\left(\lfloor \frac{\delta}{k} \rfloor + 2\right)}{2} \quad \text{and} \quad \ell_2 = \frac{\left(\lfloor \frac{\delta}{k} \rfloor + 2\right)\left(\lfloor \frac{\delta}{k} \rfloor + 3\right)}{2}.$$

For a sufficient large field $\mathbb{F}$ consider matrices

$$\mathcal{G}_r = \begin{cases} \begin{bmatrix} \boldsymbol{g}_0^{(r)} & \boldsymbol{g}_1^{(r)} & \cdots & \boldsymbol{g}_{\ell_1-1}^{(r)} & \boldsymbol{g}_{\ell_1}^{(r)} & \cdots & \boldsymbol{g}_{\ell_2-1}^{(r)} \end{bmatrix}, \text{for } r = 1, 2, \ldots, k-t \\ \begin{bmatrix} \boldsymbol{g}_0^{(r)} & \boldsymbol{g}_1^{(r)} & \cdots & \boldsymbol{g}_{\ell_1-1}^{(r)} \end{bmatrix}, \text{for } r = k-t+1, k-t+2, \ldots, k \end{cases}$$

with $\boldsymbol{g}_i^{(r)} \in \mathbb{F}^n$ for all i and r, such that

$$\mathcal{G} = \begin{bmatrix} \mathcal{G}_1 & \mathcal{G}_2 & \cdots & \mathcal{G}_{k-t} & \mathcal{G}_{k-t+1} & \cdots & \mathcal{G}_k \end{bmatrix} \tag{3}$$

is a superregular matrix and that

$$\bar{\mathcal{G}} = \begin{bmatrix} \bar{\mathcal{G}}_1 & \bar{\mathcal{G}}_2 & \cdots & \bar{\mathcal{G}}_{k-t} & \begin{matrix} \bar{\mathcal{G}}_{k-t+1} \\ \mathbf{0} \end{matrix} & \begin{matrix} \bar{\mathcal{G}}_{k-t+2} \\ \mathbf{0} \end{matrix} & \cdots & \begin{matrix} \bar{\mathcal{G}}_k \\ \mathbf{0} \end{matrix} \end{bmatrix} \tag{4}$$

is a nontrivial superregular matrix. Let $\mu : \mathbb{N}^2 \to \mathbb{N}$ be the map defined by

$$\mu(i,j) = j + \frac{(i+j)(i+j+1)}{2}, \quad \text{for all} \quad (i,j) \in \mathbb{N}^2$$

and consider the matrix

$$G(z_1, z_2) = \begin{bmatrix} G_1(z_1, z_2) & G_2(z_1, z_2) & \cdots & G_k(z_1, z_2) \end{bmatrix} \in \mathbb{F}[z_1, z_2]^{n \times k} \tag{5}$$

where

$$G_r(z_1, z_2) = \begin{cases} \displaystyle\sum_{0 \le i+j \le \lfloor \frac{\delta}{k} \rfloor + 1} \boldsymbol{g}_{\mu(i,j)}^{(r)} z_1^i z_2^j, & \text{for } r = 1, 2, \ldots, k-t \\ \displaystyle\sum_{0 \le i+j \le \lfloor \frac{\delta}{k} \rfloor} \boldsymbol{g}_{\mu(i,j)}^{(r)} z_1^i z_2^j, & \text{for } r = k-t+1, k-t+2, \ldots, k \end{cases}$$

The following theorem shows that, in case $n \ge \ell_2 k$, the above matrix $G(z_1, z_2)$ generates an MDS 2D convolutional code.

Theorem 2: *Let n, k and δ be nonnegative integers such that $n \ge \ell_2 k$ and $\mathcal{G}$, $\bar{\mathcal{G}}$ and $G(z_1, z_2)$ as defined in (3), (4) and (5), respectively, over a finite field $\mathbb{F}$. Then $\mathcal{C} = \mathrm{Im}_{\mathbb{F}[z_1,z_2]} G(z_1, z_2)$ is an MDS 2D convolutional code of rate k/n and degree δ.*

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