MORSE DECOMPOSITIONS OF SEMIFLOWS ASSOCIATED WITH GRAPHS

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EXTENDED ABSTRACT. The mathematical theory of dynamical systems analyzes, from an axiomatic point of view, the common features of many models that describe the behavior of systems in time. In its abstract form, a dynamical system is given by a time set T (with semigroup operation $\circ$), a state space M, and a map $\Phi : T \times M \to M$ that satisfies (i) $\Phi(0, x) = x$ for all $x \in M$, describing the initial value, and (ii) $\Phi(t \circ s, x) = \Phi(t, \Phi(s, x))$ for all $t, s \in T$ and $x \in M$. Common examples for the time set T are the natural numbers N or the nonnegative reals R^+ as semigroups, and the integers Z or the reals R as groups (under addition). If the state space M carries an additional structure, such a being a measurable space, a topological space or a manifold, the map Φ is required to respect this structure, i.e. it is assumed to be measurable, continuous, or differentiable, respectively.

At the heart of the theory of dynamical systems is the study of system behavior as $t \to \infty$ or $t \to \pm\infty$ (qualitative behavior), as well the change in behavior under variation of parameters (bifurcation theory).

In this note we discuss the qualitative behavior of a class of discrete mathematical models from the point of view of global systems behavior. The discrete models include finite directed graphs, certain linear iterated function systems, and Markov chains. The dynamical description of these models requires a discrete time set $T = N$, and a state space M endowed with the discrete topology, leading to a discrete semiflow $\Phi : N \times \mathcal{P}(S) \to \mathcal{P}(S)$, where $\mathcal{P}(S)$ is the power set of some finite set S. As it turns out, both the discrete topology and the fact that we are dealing with a system on a one-sided time set lead to interesting complications, when we try to adapt the ideas of chain recurrence, attractors, and Morse decompositions to our setup. We analyze the communication structure of finite directed graphs, i.e. equivalence classes of vertices that can be reached mutually via sequences of edges, and the associated quotient graphs. This leads to the 'communicating classes' $\mathcal{C} = \{C_1, ..., C_k\}$ of a graph and a reachability order $\preceq$ on $\mathcal{C}$. The key concept is that of an $L-$graph, i.e. graphs for which each vertex has out-degree ≥ 1. As it turns out, these are exactly the graphs for which the $\omega-$limit sets of the associated semiflow are nonempty. To each graph $G = (V, E)$, where V is the set of vertices and $E \subset V \times V$ the set of edges, we associate a semiflow $\Phi_G : N \times \mathcal{P}(V) \to \mathcal{P}(V)$. This semiflow is studied from the point of view of qualitative behavior of dynamical systems, i.e. we adapt the concepts of $\omega-$limit sets, (positive) invariance, recurrence, Morse decompositions, attractors and attractor-repeller pairs to Φ_G and prove characterizations equivalent to those given in the standard literature. As it turns out, the finest Morse decomposition of Φ_G corresponds to the decomposition $\mathcal{C}$ of the graph G into communicating classes, and the order on the communicating classes is equivalent to the order that accompanies a Morse decomposition. Moreover, the connected components of the recurrent set of Φ_G are exactly the (finest) Morse sets of Φ_G, i.e. the communicating classes of G.

1. DEFINITIONS AND MAIN RESULTS. Let $G = (V, E)$ be a finite, simple directed graph with positive outdegree for each element of V, and let Φ_G be the associated semiflow to G. For a detailed discussion on the definition of the semiflow associated to G and related constructions refer to [5].

DEFINITION 1.1. *A point $A \in \mathcal{S} = \mathcal{P}(V)$ is said to be weakly invariant if for all $n \in N$ we have $\Phi_G(n, A) \cap A \neq \emptyset$.*

DEFINITION 1.2. Consider the semiflow Φ_G and a finite collection $\mathcal{A} = \{A_1, ..., A_n\}$ of points in $\mathcal{S}$. $\mathcal{A}$ is said to satisfy the no-cycle condition for Φ_G if for any subcollection $A_{j_0}, ..., A_{j_l}$ of $\mathcal{A}$ with $\omega(A_{j_\alpha}) \cap A_{j_{\alpha+1}} \neq \emptyset$ for $\alpha = 0, ..., l-1$ it holds that $A_{j_0} \neq A_{j_l}$.

We denote by $\mathcal{R}$ to the recurrent set associated with the given semiflow.

DEFINITION 1.3. Let $G = (V, E)$ be an $L-$graph. A Morse decomposition of the semiflow Φ_G on $\mathcal{S} = \mathcal{P}(V)$ is a finite collection of nonempty, pairwise disjoint and weakly invariant sets $\{\mathcal{M}_\mu \in \mathcal{S} : \mu = 1, ..., k\}$ such that:

(a) $\mathcal{R} \subset \cup_{\mu=1}^{k} \mathcal{M}_\mu$

(b) $\{\mathcal{M}_\mu \in \mathcal{S} : \mu = 1, ..., k\}$ satisfies the no-cycle condition.

The main results of this work are listed bellow:

1. Characterization of Morse Decompositions via the order induced by the semiflow:

   PROPOSITION 1.4. Let $G = (V, E)$ be an $L-$graph and let,

   $$\mathcal{M} = \{\mathcal{M}_\mu \in \mathcal{S} : \mu = 1, ..., k\}$$

   be a finite collection of nonempty, pairwise disjoint and **weakly invariant** sets of the semiflow Φ_G on $\mathcal{S} = \mathcal{P}(V)$. The collection $\mathcal{M}$ is a Morse decomposition of Φ_G iff the following properties hold: (i) $\mathcal{R} \subset \cup_{\mu=1}^{k} \mathcal{M}_\mu$ and (ii) the relation "$\preceq$" defined by

   $$\mathcal{M}_\alpha \preceq \mathcal{M}_\beta \text{ if } \begin{array}{c} \text{there are } \mathcal{M}_{j_0} = \mathcal{M}_\alpha, \mathcal{M}_{j_1}, ...\mathcal{M}_{j_l} = \mathcal{M}_\beta \text{ in } \mathcal{M} \\ \text{with } \omega(\mathcal{M}_{j_i}) \cap \mathcal{M}_{j_{i+1}} \neq \emptyset \text{ for } i = 0, ..., l-1 \end{array}$$

   is a (partial) order on $\mathcal{M}$.

   We use the indices $\mu = 1, ..., k$ in such a way that they reflect this order, i.e. if $\mathcal{M}_\alpha \preceq \mathcal{M}_\beta$ then $\alpha \leq \beta$.
2. Characterization of the finest Morse Decomposition for the semiflow associated with a $L-$graph:

   THEOREM 1.5. Let $G = (V, E)$ be an $L-$graph with associated semiflow Φ_G on $\mathcal{S} = \mathcal{P}(V)$. For a finite collection of nonempty, pairwise disjoint sets $\mathcal{M} = \{\mathcal{M}_\mu \in \mathcal{S} : \mu = 1, ..., k\}$ the following statements are equivalent:

   (a) $\mathcal{M}$ is the finest Morse decomposition of Φ_G.

   (b) $\mathcal{M} = \mathcal{C}$, the set of communicating classes of G.
3. Characterization of Morse Decompositions of Semiflows via attractor-repellers pairs, in analogy to [4]:

   THEOREM 1.6. Let $G = (V, E)$ be an $L-$graph with associated semiflow Φ_G on $\mathcal{S} = \mathcal{P}(V)$. A finite collection of sets $\mathcal{M} = \{\mathcal{M}_\mu \in \mathcal{S} : \mu = 1, ..., k\}$ defines a Morse decomposition of Φ_G if and only if there is a strictly increasing sequence of attractors

   $$\emptyset = A_0 \subset A_1 \subset A_2 \subset ... \subset A_n \subset \mathcal{S},$$

   such that

   $$\mathcal{M}_{n-i} = A_{i+1} \cap A_i^* \text{ for } 0 \leq i \leq n-1.$$

4. Characterization of the semiflow Φ_G on its Morse sets:
   THEOREM 1.7. Let $G = (V, E)$ be an $L-$graph with associated semiflow Φ_G on $\mathcal{S} = \mathcal{P}(V)$. The recurrent set $\mathcal{R}$ of Φ_G satisfies

$$\mathcal{R} = \bigcap \{A \cup A^*, \ \ A \text{ is an attractor}\}$$

   and the (finest) Morse sets of Φ_G coincide with the Φ_G-connected components of $\mathcal{R}$.

In addition, given an $L-$graph $G = (V, E)$, with associated adjacency matrix A, we construct a semiflow $\Psi_A : N \times Q \to Q$ (where Q is the unit cube in R^n) that is equivalent to the semiflow Φ_G defined on $\mathcal{P}(V)$, using logical matrix multiplication. This point of view is somewhat different from the standard approach that uses regular matrix multiplication and that does not lead to an equivalent semiflow. The equivalence allows us to interpret all results obtained for Φ_G in terms of certain linear iterated function systems.

We apply the results obtained for graphs and their semiflows to the study of general finite Markov chains. Our presentation unifies many of the concepts and shows which structural (deterministic, graph theoretic, semiflow) properties and which probabilistic properties are really needed to analyze Markov chains.

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