A GEOMETRIC APPROACH TO SHORTEST BOUNDED CURVATURE PATHS IN SURFACES OF CONSTANT NON-POSITIVE CURVATURE

JOSE AYALA-HOFFMANN, HYAM RUBINSTEIN, DOREEN THOMAS, MARCUS BRAZIL

Abstract. Consider $(x, X), (y, Y) \in T\mathbb{M}$, where $\mathbb{M}$ corresponds to the hyperbolic or Euclidean plane $\mathbb{H}$ or $\mathbb{R}^2$ and $T\mathbb{M}$ is the unit tangent bundle. In this work we address the problem of characterizing the paths of bounded curvature and minimal length starting at x, finishing at y and having tangents at these points X and Y respectively. This problem was first investigated in the late 1950's for the case $T\mathbb{R}^2$ by Lester Dubins. In this note we present a constructive solution giving a special emphasis on the geometric nature of this problem.

Key words. Dubins problem, surfaces of non-positive curvature

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1. Introduction. Let us consider $\mathbb{M}$ as $\mathbb{R}^2$, or the hyperbolic plane $\mathbb{H}$.

DEFINITION 1.1. Given $(x, X), (y, Y) \in T\mathbb{M}$, we say that a path $\alpha : [0, s] \to \mathbb{M}$ connecting these points is admissible if:

1. α is C^1, and piecewise C^2
2. α is parametrized by arc length (i.e $|\alpha'(t)| = 1$ for all $t \in [0, s]$)
3. $\alpha(0) = x$, $\alpha'(0) = X$; $\alpha(s) = y$, $\alpha'(s) = Y$
4. $||\alpha''(t)|| \leq k$, for all $t \in [0, s]$ when defined, $k > 0$ a constant.

Here s is the arc-length of α.

Condition (1) means that α has continuous first derivative and piecewise continuous second derivative. For condition (3), without loss of generality, we can extend the domain of α to $(-\epsilon, s+\epsilon)$ for ϵ arbitrarily small. Sometimes we describe condition (3) as a boundary condition. Condition (4) means that α has absolute curvature bounded above by a positive constant. We denote the interval $[0, s]$ by I.

DEFINITION 1.2. *The space of embedded admissible paths for fixed boundary conditions* $(x, X), (y, Y) \in T\mathbb{M}$ *is denoted by* $\mathcal{E}$.

It is important to notice that the topological and geometrical properties of $\mathcal{E}$ depend on the choice of elements in $T\mathbb{M}$.

We now state the problem that motivates this work: Given $(x, X), (y, Y) \in T\mathbb{M}$, and a maximum curvature $k > 0$, find an embedded admissible path of minimal length satisfying the given boundary conditions.

2. Path Normalization. Given $(x, X), (y, Y) \in T\mathbb{M}$, we are interested in finding a path of minimum length in $\mathcal{E}$. In the case of the Euclidean plane, by rescaling we can always assume that the maximum curvature $k = 1$. We introduce a *path normalization* so that only paths of type *cs* (paths that corresponds to multiple concatenations of arcs of a circle of unit radius and length less than 2π, followed by line segments) need be considered. In this context we present the following definition.

For an admissible path $\alpha : I \to \mathbb{R}^2$ we denote by $\mathcal{L}(\alpha, a, b)$ the length of α between $[a, b] \subset I$.

DEFINITION 2.1. *A π-fragmentation of an admissible path $\alpha : I \to \mathbb{R}^2$ corresponds to a finite sequence $0 = t_0 < t_1 \ldots < t_n = s$ of elements in I such that,*

$$\mathcal{L}(\alpha, t_{i-1}, t_i) < \pi$$

with,

$$\sum_{i=1}^{n} \mathcal{L}(\alpha, t_{i-1}, t_i) = s$$

LEMMA 2.2. *Suppose that $\alpha \in \mathcal{E}$ so that $\alpha(0) = (0,0)$, $\alpha'(0) = (1,0)$ and the length of α satisfies $s < \pi r$. Then the length of β is at most the length of α with equality if and only if $\alpha = \beta$.*

Remark: This key result is proved by a direct projection argument. First of all, it is easy to construct the path β with the same boundary conditions as α. Then the idea is to divide α up into at most three pieces and project each one onto the corresponding piece (circular arc or straight line segment) of β, so that the length of the piece of α does not increase under the projection.

DEFINITION 2.3. *An admissible path is in cs form if it is a finite number of concatenations of line segments, or arcs of unit circles of length $< 2\pi$. The number of line segments and circular arcs is called the complexity of the path.*

Remark: Observe that starting with a given admissible path α, we can achieve many *cs* paths with the same boundary conditions. In fact, such paths will depend on the π-fragmentation of α that we choose to apply lemma 2.2. Moreover the *cs* paths have length at most that of our original path α. We refer to this process as *normalization* of α.

3. Path Replacements. In the previous section we normalized an arbitrary embedded admissible path α (without changing the boundary conditions) to a *cs* path of length at most the length of α. In this section we continue our simplification procedure for admissible paths. The next process provides a way to replace an arbitrary *cs* path of complexity n (without changing the boundary conditions) by a *cs* path of complexity at most 3 and length not superior to the length of the complexity n *cs* path.

DEFINITION 3.1.

1. *We denote by $\mathfrak{C}_1$ a component of type csccsc shown in Figure 3.1*
2. *We denote $\mathfrak{C}_2$ a component of type cscsc shown in Figure 3.2*

Remark: In both cases, the complexity could be four or five, since one or two of the circular or straight arcs could have length zero.

PROPOSITION 3.2. *Every cs path of complexity at least 4 has such a component $\mathfrak{C}_1$ and $\mathfrak{C}_2$.*

THEOREM 3.3. *Given an admissible path α, there is a cs path of complexity at most 3 with the same boundary conditions and length not more than the length of α.*

Towards the main result of this note (Theorem 3.4), we continue the simplification process via two results. These provide a way to replace an arbitrary *cs* paths by a *cs* path of complexity at most 3.

THEOREM 3.4. *A cs path with a component of type $\mathfrak{C}_1$ can be replaced by a cs path with less complexity, and without increasing its length.*

COROLLARY 3.5. *A cs path with a component of type $\mathfrak{C}_2$ can be replaced by a cs path with less complexity, and without increasing its length.*

Remark: The points x_i in Figure 3.1 and Figure 3.2, are such that $\alpha(t_i) = x_i$, where the t_i are elements in some π-fragmentation of α, with $i = 1, \ldots, n$.

PROPOSITION 3.6. *cs paths having complexity n, where $n > 3$, are not (global) minima of length in the class of admissible paths with fixed boundary conditions.*

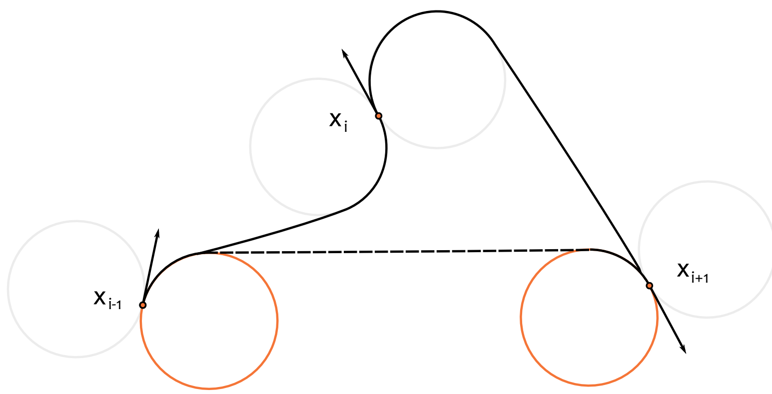

FIG. 3.1. *A concatenated component of type $\mathfrak{C}_1$. The dashed line correspond a replacement by a csc path.*

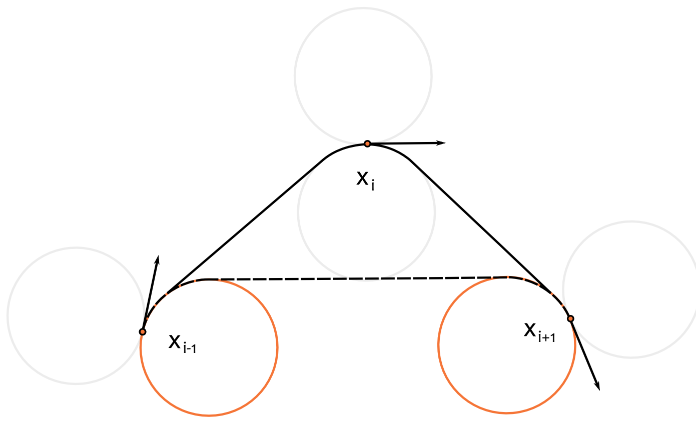

FIG. 3.2. *A concatenated component of type $\mathfrak{C}_2$. The dashed line correspond a replacement by a csc path.*

THEOREM 3.7. *Given $(x, X), (y, Y) \in T\mathbb{M}$, and a maximum curvature $k > 0$, then the admissible path of minimal length satisfying the given boundary conditions is of form csc, or ccc (where some of the circular arcs or line segments can have zero length).*

4. Remarks on the Hyperbolic case. In the hyperbolic case, lemma 2.2 is a little more complicated to prove. But otherwise exactly the same approach works. Paths of constant curvature k are arcs of circles in the hyperbolic plane. Note that in this case, we cannot rescale to only deal with $k = 1$ since rescaling changes the underlying curvature. So we need to always work with arcs of circles of appropriate fixed radius depending on the choice of k.

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