PRACTICAL DECODERS FOR BARNES-WALL LATTICE CONSTELLATIONS

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EXTENDED ABSTRACT. In 2008, Micciancio and Nicolosi [1] have proposed a low-complexity algorithm to decode the infinite Barnes-Wall lattice \(BW_{2m} \subset \mathbb{C}^{2m}\) for any \(m \geq 1\) such that the worst-case complexity is \(O(N \log^2(N))\) where \(N = 2^m\). More recently, Construction \(A'\) [2] of Barnes-Wall lattices has been proposed wherein the \(BW_{2m}\) can be obtained as \(BW_{2m} = (1+i)^m \mathbb{Z}[i]^{2m} \oplus L_{2m}\) such that \(L_{2m}\) is a cubic-shaped lattice constellation.

Inspired by the existence of the low-complexity Barnes-Wall lattice decoder, in this paper, we study the error performance of \(L_{2m}\) as a coded modulation scheme for AWGN channels. To encode the code, we use Construction \(A'\), and to decode the code we use the infinite Barnes-Wall lattice decoder (IBWD) [1]. First, we study the error performance of IBWD in decoding the infinite lattice and then propose a variant of it called the Barnes-Wall lattice constellation decoder (BWCD) to decode the lattice constellation. We also present simulation results on the bit error rate of the decoders.

Main results. In this paper, we study the error-performance of Barnes-Wall (BW) lattice [3], [4] constellations in AWGN channels. Throughout the paper, we use \(BW_N\) to denote the BW lattice in \(N = 2^m\) complex dimensions for any \(m \geq 1\).

We employ the cubic shaped constellation \(L_N \subset BW_N\) [2] as a coded modulation scheme in AWGN channels. At the transmitter, we use Construction \(A'\) for encoding information bits [2], and at the receiver, we use the IBWD [1] to decode the code.

First, we study the error performance of the IBWD algorithm. It is shown in [1] that for any \(x \in BW_N\), if \(y \in \mathbb{C}^N\) such that

\[
d_{\text{min}}^2(x, y) < \frac{N}{4},
\]

then the IBWD correctly finds (or decodes) the closest lattice point \(\hat{x} = x\). In the context of using IBWD for AWGN channels, the vector \(y\) corresponds to \(y = x + n\), where \(x \in BW_N\) and \(n \sim \mathcal{CN}(0, \sigma^2)\). For the AWGN channel, (0.1) implies that the codeword error rate (CER) of the IBWD (denoted by \(P(\hat{x} \neq x)\)) is bounded as

\[
P(\hat{x} \neq x) \leq P \left( |n|^2 > \frac{N}{4} \right).
\]

Note that \(\sqrt{\frac{N}{2}}\) is the minimum Euclidean distance of \(BW_N\) and hence, (0.2) provides the well known sphere-upper bound [6]. In [1], no comment is made on the tightness.

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of this upper bound, i.e., the possibility of correct decision even when $|n|^2 > \frac{N}{4}$. To know the explicit error performance of the IBWD, we analyse the IBWD algorithm, and subsequently find the tightness of the sphere upper bound.

The IBWD algorithm is a successive interference cancellation (SIC) type decoder which exploits the BW lattice structure as a multi-level code of nested Reed-Muller (RM) codes (as per Construction $D$). At each level, the algorithm uses a soft-decision RM decoder proposed in [5] to estimate and subtract the RM codeword in that level. Therefore, the error performance of the IBWD is determined by the error performance of the underlying soft-input RM decoder at each level as,

$$P(\hat{x} \neq x) = P \left( \bigcup_{r=0,1,\cdots,m-1} E(\hat{c}_r \neq c_r) \right), \quad (0.3)$$

where $E(c_r \neq c_r)$ is the event of incorrect decoding in the $r$-th order RM code. Now, we study the CER of the soft-input RM decoder (denoted by $P(\hat{c}_r \neq c_r)$) used in the IBWD. The RM codewords at each level of the BW lattice take values from the alphabet $\{0, 1\}$. Therefore, to decode the RM code at each level, a hard-decision binary vector $b$ of the form,

$$b = [\Re(y)] + [\Im(y)] \text{ mod } 2 \quad (0.4)$$

is obtained from the received vector $y$, where $[\cdot]$ denotes the nearest integer operator. Also, the soft-input metric passed to the RM decoder is given by $\rho = 1 - 2d$, where $d = \max(|\Re(y)| - \Re(y)|, |\Im(y)| - \Im(y)|)$. Unlike the soft metric in [5], in this case, $\rho_j$ is bounded in the interval $[0, 1]$. This is because $d_j \in [0, 0.5]$, which is a result of the mod operation. We could imagine $b$ and $\rho$ to be obtained from a received vector $\tilde{y}$ in a virtual additive noise channel, wherein each component of $\tilde{y}$ is always within a $L_1$-distance of 0.5 from the symbols of the code alphabet $\{0, 1\}$. Therefore, if $c \in \{0, 1\}^N$ denotes a RM codeword at a particular level of the transmitted BW lattice point, then the effective noise $n_{eff}$ as seen by the soft-input RM decoder is of the form,

$$n_{eff}^j = \begin{cases} d_j, & \text{when } b_j = c_j \\ 1 - d_j, & \text{when } b_j \neq c_j, \end{cases} \quad (0.5)$$

for $1 \leq j \leq N$. Note that $n_{eff}^j$ has bounded support in the interval $[0, 1]$. For an analogy with respect to the model in [5], the code alphabet $\{0, 1\}$ in [1] corresponds to the code alphabet $\{-1, 1\}$ in [5] and the effective noise $n_{eff}^j$ in [1] corresponds to the AWGN in [5]. At each level of the BW lattice, the Euclidean code $(1 + i)^rR_M(r, m)$ for any $0 \leq r \leq m - 1$ has the minimum squared Euclidean distance of $N$. Therefore, by using the proposition in Section IV.A of [5], the probability of incorrect decision of the soft-input RM decoder at each level of IBWD is upper bounded as,

$$P(\hat{c}_r \neq c_r) \leq P \left( |n_{eff}^j|^2 > \frac{N}{4} \right) \quad (0.6)$$

for $r = 0, 1, \cdots, m - 1$. It is important to note that the above bound is different from $P(|n|^2 > \frac{N}{4})$ since $n$ is Gaussian distributed. In Fig. 0.1, we display the histogram of the realizations of $n_{eff}^j$ for different variance values of $n_j$, when the zero RM codeword is the transmitted.

We now present the CER of the IBWD through simulation results. In Fig. 0.2, we present the CER of the IBWD for BW lattice of complex dimension 1024. In the
same plot, we also present (i) the sphere upper bound (SUB) given by $P(|\mathbf{m}|^2 > \frac{N}{4})$, (ii) the sphere lower bound (SLB) [6], (iii) the CER in decoding $\mathcal{R}M(0, m)$ at the first level of the IBWD which is $P(c_0 \neq c_0)$, and (iv) an upper bound on the CER in decoding $\mathcal{R}M(0, m)$ at the first level of the IBWD, given by $P(|\mathbf{n}^{eff}|^2 > \frac{N}{4})$. Simulation results for other dimensions of BW lattices can be found in [7]. From Fig. 0.2, we make the following observations: The sphere upper bound is not a tight upper bound on the CER of IBWD for large dimensions. Also, $P(|\mathbf{n}^{eff}|^2 > \frac{N}{4})$ is an upper bound on the CER of IBWD and in particular, it is a tighter upper bound than the sphere upper bound. For larger dimensions, the sphere lower bound [6] is quite far from the CER of IBWD, which hints that the IBWD performance is quite poor and far from that of the ML decoder. The CER of the soft-input RM decoder for $\mathcal{R}M(0, m)$ is a tight lower bound on the CER of the IBWD. This implies that if there is no error in the first level of the decoder, then with high probability, there will be no errors in the further levels of the soft-input RM decoder. Similar observations also hold true for other dimensions as well [7].

Till now, we have discussed the error performance of the IBWD in decoding infinite BW lattice. We now discuss the use of IBWD to decode $\mathcal{L}_N$. When a codeword
of $\mathcal{L}_N$ is transmitted, the IBWD decodes to a lattice point in the infinite lattice $BW_N$. In such a decoding method, irrespective of whether the decoded lattice point falls in the code, the information bits can be recovered from the decoded RM codewords at every level of IBWD. To overcome this sub-optimality of IBWD to decode $\mathcal{L}_N$, we use a noise trimming technique that forces the IBWD to decode to a codeword in the Euclidean code $\mathcal{L}_N$. We refer to such a decoder as a BW lattice constellation decoder (BWCD). From the cubic shaping of $\mathcal{L}_N$, we know that each component of the codeword is within a rectangular box $B$. In order to use IBWD, and to decode to a codeword within the code, we trim the in-phase and quadrature components of the received vector (the full algorithm is given in [7]) to be within a box $B' \supseteq B$ marginally larger than $B$ by length $\epsilon$ on each dimension. After the trimming technique, we feed the trimmed received vector to the IBWD and decode the information bits. Note that the choice of $\epsilon$ is crucial to decode a codeword within the code, and to improve the BER with reference to the IBWD. Using BWCD, we have obtained BER for dimensions when $m = 2, 4$, and $6$, and compared them with the BER of the IBWD. The plots as shown in Fig. 0.3 indicate that BWCD outperforms IBWD by 0.5 dB. For the presented results, we have used $\epsilon = \frac{1}{2\sqrt{2}}$.

![BER comparison between BWCD and IBWD](image)

Fig. 0.3. Comparison of BER between BWCD and IBWD for $m = 2, 4$, and 6.

REFERENCES


