STABILIZATION AND CONTROL DESIGN BY PARTIAL OUTPUT FEEDBACK

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Abstract. We consider a (not necessarily proper) plant input/output behavior which shall be stabilized by a proper controller via partial output feedback such that also the feedback behavior is a proper input/output behavior. In addition a control task like tracking or disturbance rejection shall be performed. The setting includes both the continuous and the discrete time case, and stability is defined with respect to a set T of stable polynomials. The standard choice for T yields asymptotic stability.

Our approach relies on the fact that the signal module is an injective cogenerator over the ring of operators and this property is preserved under localization with respect to the set T.

We present a condition ensuring the existence of proper compensators such that the feedback behavior is proper and stable and such that the given control task is performed. If this condition is satisfied we construct a large class of such compensators.

Introduction. Questions of stabilization and control have always been central problems of systems theory and are extensively treated in the literature. We refer to the introduction of [2] for a long list of prominent researchers and books in this area of systems theory. For brevity we only mention the seminal contributions of Kučera [4] and Youla et al. [11, 12] on parametrization of stabilizing compensators and of Vidyasagar [9] on principal ideal domains of proper stable rational functions which also play an important part in [2] and the present article.

We use the behavioral approach and consider behaviors whose signal components belong to an injective cogenerator signal module $\mathscr{F}$ over a polynomial algebra $F[s]$ over an infinite field F as ring of operators with action $\circ$. The standard continuous or discrete one-dimensional linear time-invariant systems are included in this framework. We define stability w.r.t. a saturated multiplicatively closed set T of nonzero polynomials in $F[s]$, called *T-stable polynomials*: A signal is called *T-small* or *T-negligible* if it is annihilated by some polynomial in T, a behavior is *T-autonomous* if all its trajectories are T-small, and an input/output behavior is *T-stable* if its autonomous part is T-autonomous. If $\Omega \subseteq \overline{F}$ is the set of all roots of polynomials in T in an algebraic closure $\overline{F}$ of F, for instance $F = \mathbb{R} \subseteq \overline{\mathbb{R}} = \mathbb{C}$, then T consists of all polynomials with roots in Ω, compare [9, p.14f], [8, Ch.5]. We use both the rings $F[s]_T$ of *T-stable rational functions* and their subrings $\mathscr{S}$ of *proper and T-stable* rational functions [9, Ch.2], [8, Ch.5].

We consider the partial output feedback interconnections displayed in Figure 1 which were suggested by Willems at the MTNS 2010 conference in Budapest. This configuration is a generalization of [10, Eq.11], the two-parameter compensators from [9, Sec.5.6], and the RST-controllers from [3, Ch.7]. Both the plant $\mathscr{B}_1$ and the compensator $\mathscr{B}_2$ are input/output (IO) behaviors and the interconnected system is required to have the following properties: (i) The feedback system is well-posed, i.e., is itself an IO behavior, and proper and T-stable, i.e., its autonomous part is T-autonomous and its transfer matrix is proper and (automatically) T-stable. (ii) The compensator $\mathscr{B}_2$ is proper. Up to the properness of $\mathscr{B}_1$ which is *not* required here the feedback interconnection is *regular* in the sense of [10, p.334]. (iii) The interconnected system performs desired tasks like tracking or disturbance rejection. More precisely: If the inputs u_1 and u_2 of plant and compensator are contained in given external behaviors then a given linear function of the signals of plant and compensator produces T-small signals.

For a given plant $\mathscr{B}_1$ and the desired properties (i)-(iii) the problem is is to *decide the existence* of such compensators $\mathscr{B}_2$, to *parametrize all* of them and also to solve the problem

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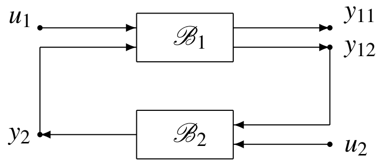

FIG. 0.1. *The partial output feedback behavior* $\mathrm{fb_p}(\mathscr{B}_1, \mathscr{B}_2)$.

of *pole placement* of the feedback system. We solve this problem constructively, but we do not construct all, but only a large class of such compensators, the properness of $\mathscr{B}_2$ requiring special attention and special new algorithms. The references [3], [9], and [10] mentioned above give partial solutions to the described problem.

We use a technique which has been introduced by Oberst and which has already been successfully applied in [2] and the thesis [1] and which depends essentially on the fact that the quotient module $\mathscr{F}_T$ is an injective cogenerator over the ring $F[s]_T$. In particular, the action of $h \in F[s]_T$ on $u \in \mathscr{F}_T$ gives a well-defined trajectory $h \circ u \in \mathscr{F}_T$ whereas the often used hu, for instance for $h \in \mathbb{R}(s) \setminus \mathbb{R}[s]$ and $u \in \mathscr{F} := \mathscr{C}^\infty(\mathbb{R}, \mathbb{R})$, is defined for the Laplace transforms (i.e., in the frequency domain) only and not sufficient when actual behaviors and not only their transfer matrices are studied. More generally, every stable rational transfer matrix $H \in F[s]_T^{p \times m}$ induces an *IO-operator* $H\circ : \mathscr{F}_T^m \to \mathscr{F}_T^p$.

Stabilization and regulation have recently been studied for more general regular interconnections of behaviors instead of the partial feedback interconnections of the present paper by Belur, Fiaz, Praagman, Rocha, Takaba, Trentelman, Willems, Zavala, and others. In the author's thesis [1] the technique of the present article has also been successfully applied to these interconnections; the results, as well as the proofs of the results of the present paper, will appear in a forthcoming publication.

1. Preliminaries. We consider the ring of operators $\mathscr{D} := F[s]$, i.e., the polynomial ring over some infinite field F in one indeterminate s, and the signal module $\mathscr{F}$ which is assumed to be an injective cogenerator over $\mathscr{D}$. A behavior is a set of the form $\mathscr{B} = \{w \in \mathscr{F}^\ell; R \circ w = 0\}$ where $R \in \mathscr{D}^{k \times \ell}$, i.e., the set of solutions of a system of linear time-invariant homogeneous differential resp. difference equations in the continuous resp. discrete standard case. We summarize some results on localization of $\mathscr{D}$ and $\mathscr{F}$ with respect to the multiplicatively closed saturated set $T \subseteq \mathscr{D} \setminus \{0\}$ which have already been used in [2]. The set T gives rise to the quotient ring $\mathscr{D}_T := \left\{ \frac{d}{t} \in F(s); d \in \mathscr{D}, t \in T \right\}$ of $\mathscr{D}$ w.r.t. T and to the quotient signal module $\mathscr{F}_T := \left\{ \frac{w}{t}; w \in \mathscr{F}, t \in T \right\}$. Elements of $\mathscr{D}_T$ are called *T-stable rational functions*. The quotient or localization functor $(-)_T$ is exact. $\mathscr{F}_T$ is an injective cogenerator over $\mathscr{D}_T$, and the signal module $\mathscr{F}$ is isomorphic to $\mathscr{F}_T \oplus \mathrm{t}_T(\mathscr{F})$ where $\mathrm{t}_T(\mathscr{F}) := \{w \in \mathscr{F}; \exists t \in T : t \circ w = 0\}$ denotes the T-torsion module of $\mathscr{F}$, i.e., the set of all T-small signals in $\mathscr{F}$. Hence elements of $\mathscr{F}_T$ can be interpreted as the "essential part" of signals in $\mathscr{F}$, or as signals in $\mathscr{F}$ "up to a T-small (and hence negligible) part". Likewise, a behavior $\mathscr{B} = \{w \in \mathscr{F}^\ell; R \circ w = 0\}$, $R \in \mathscr{D}^{k \times \ell}$, can be decomposed as $\mathscr{B} \cong \mathscr{B}_T \oplus \mathrm{t}_T(\mathscr{B})$ with $\mathrm{t}_T(\mathscr{B}) := \mathscr{B} \cap (\mathrm{t}_T(\mathscr{F}))^\ell$. Again $\mathscr{B}_T$ can be interpreted as $\mathscr{B}$ "up to a T-small part". Moreover, $\mathscr{B}_T = \{w \in \mathscr{F}_T^\ell; R \circ w = 0\}$ and $(P \circ \mathscr{B})_T = P \circ \mathscr{B}_T$ for $P \in \mathscr{D}^{p \times \ell}$.

DEFINITION 1.1 (T-autonomy, T-stability, T-observability).

1. *A behavior* $\mathscr{B} = \left\{w \in \mathscr{F}^\ell; R \circ w = 0\right\}$, $R \in \mathscr{D}^{k \times \ell}$, *is called* T-autonomous *if all its signals are T-small, i.e., if* $\mathscr{B} \subseteq (\mathrm{t}_T(\mathscr{F}))^\ell$. *By the direct sum decomposition of behaviors this is the case iff* $\mathscr{B}_T = 0$ *or, since* $\mathscr{F}_T$ *is an injective cogenerator over* $\mathscr{D}_T$, *iff R admits a left inverse in* $\mathscr{D}_T^{\ell \times k}$. *$T$-autonomy signifies being zero resp. autonomy resp. autonomy and*

asymptotic stability for standard choices of T.

2. *An input/output behavior [6, Sec.3.3], [5, Thm.2.69 on p.37]* $\mathscr{B}=\{\binom{y}{u}\in\mathscr{F}^{p+m};\, P\circ y=Q\circ u\}$ *with* $(P,-Q)\in\mathscr{D}^{k\times(p+m)}$, $\operatorname{rank}(P)=\operatorname{rank}(P,-Q)=p$, *is called* T-stable *if its autonomous part* $\mathscr{B}^0:=\{y\in\mathscr{F}^p;\, P\circ y=0\}$ *is T-autonomous.*

3. *For a behavior* $\mathscr{B}=\{\binom{w_r}{w_m}\in\mathscr{F}^{r+m};\, R_r\circ w_r+R_m\circ w_m=0\}$, $(R_r,R_m)\in\mathscr{D}^{k\times(r+m)}$, *$w_r$ is* T-observable *from w_m in $\mathscr{B}$ if $w_m=0$ implies that w_r is T-small for all* $\binom{w_r}{w_m}\in\mathscr{B}$. *This is the case iff R_r admits a left inverse in* $\mathscr{D}_T^{r\times k}$. *$T$-observability coincides with observability, trackability, or detectability in standard cases as introduced in [7, Def.2.1].*

When dealing with properness issues we use the ring $\mathscr{S}:=\mathscr{D}_T\cap F(s)_{\mathrm{pr}}$ of proper T-stable rational functions where $F(s)_{\mathrm{pr}}:=\left\{\frac{f}{g}\in F(s);\, f,g\in\mathscr{D}, g\neq 0, \deg(f)\leq\deg(g)\right\}$. We always assume that T contains an element $(s-\alpha)$ for some $\alpha\in F$ (otherwise saturation of T would imply that $T=\mathbb{C}\setminus\{0\}$ and hence $\mathscr{S}=\mathbb{C}$ in the standard case $F=\mathbb{C}$). Defining $\sigma:=(s-\alpha)^{-1}$ and $\widehat{\mathscr{D}}:=F[\sigma]$ yields $\mathscr{S}=\widehat{\mathscr{D}}_{\widehat{T}}$ where $\widehat{T}:=\left\{\frac{t}{(s-\alpha)^{\deg(t)}};\, t\in T\right\}\subseteq\widehat{\mathscr{D}}$, and $F(s)_{\mathrm{pr}}=\widehat{\mathscr{D}}_{\widehat{\mathscr{D}}\setminus\widehat{\mathscr{D}}\sigma}$. Moreover, the quotient ring $\mathscr{S}_\sigma:=\left\{\frac{\xi}{\sigma^j};\, \xi\in\mathscr{S},\, j\in\mathbb{N}\right\}$ is equal to $\mathscr{D}_T$.

2. Main Results. According to Figure 0.1 we consider a plant $\mathscr{B}_1$ and a compensator $\mathscr{B}_2$ where

$$\begin{aligned}
\mathscr{B}_1 &= \left\{\begin{pmatrix} y_{11}\\ y_{12}\\ y_2\\ u_1\end{pmatrix}\in\mathscr{F}^{p_1+p_{12}+m+m_1};\, (P_{11},P_{12})\circ\begin{pmatrix}y_{11}\\ y_{12}\end{pmatrix}=(Q_{12},Q_{11})\circ\begin{pmatrix}y_2\\ u_1\end{pmatrix}\right\}\\
&\quad\text{with}\quad p:=p_1+p_{12},\quad y_1:=\binom{y_{11}}{y_{12}}\in\mathscr{F}^{p_1+p_{12}}=\mathscr{F}^p,\\
&\quad P_1:=(P_{11},P_{12})\in\mathscr{D}^{p\times(p_1+p_{12})},\ \det(P_1)\neq 0,\quad (Q_{12},Q_{11})\in\mathscr{D}^{p\times(m+m_1)}, \qquad (2.1)\\
\mathscr{B}_2 &= \left\{\begin{pmatrix} u_2\\ y_{12}\\ y_2\end{pmatrix}\in\mathscr{F}^{p_2+p_{12}+m};\, P_2\circ y_2=(Q_{22},Q_{21})\circ\begin{pmatrix}u_2\\ y_{12}\end{pmatrix}\right\}\\
&\quad\text{with}\quad P_2\in\mathscr{D}^{m\times m},\ \det(P_2)\neq 0,\ (Q_{22},Q_{21})\in\mathscr{D}^{m\times(p_2+p_{12})}.
\end{aligned}$$

We will study existence and construction of compensators $\mathscr{B}_2$ such that the partial output feedback interconnection $\mathrm{fb}_{\mathrm{p}}(\mathscr{B}_1,\mathscr{B}_2)$ of $\mathscr{B}_1$ and $\mathscr{B}_2$ from Figure 0.1 is a T-stable input/output behavior with input $\binom{u_1}{u_2}$ and output $\binom{y_1}{y_2}$. We require both the compensator and the feedback behavior to have proper transfer matrix. We do however not assume properness of the plant $\mathscr{B}_1$.

DEFINITION 2.1. *Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be input/output behaviors as in* (2.1).

1. *As visualized in Figure 0.1 the* partial feedback behavior *of $\mathscr{B}_1$ and $\mathscr{B}_2$ is defined as*

$$\mathrm{fb}_{\mathrm{p}}(\mathscr{B}_1,\mathscr{B}_2):=\left\{\begin{pmatrix}y_1\\ y_2\\ u_1\\ u_2\end{pmatrix}\in\mathscr{F}^{p+m+m_1+p_2};\ \begin{pmatrix}P_1 & -Q_{12}\\ -Q_2 & P_2\end{pmatrix}\circ\begin{pmatrix}y_1\\ y_2\end{pmatrix}=\begin{pmatrix}Q_{11} & 0\\ 0 & Q_{22}\end{pmatrix}\circ\begin{pmatrix}u_1\\ u_2\end{pmatrix}\right\}$$

where $Q_2:=(Q_{20},Q_{21})\in\mathscr{D}^{m\times(p_1+p_{12})}=\mathscr{D}^{m\times p}$ *with* $Q_{20}=0\in\mathscr{D}^{m\times p_1}$. *In the notation of Willems* $\mathrm{fb}_{\mathrm{p}}(\mathscr{B}_1,\mathscr{B}_2)=\mathscr{B}_1\wedge_{(y_{12},y_2)}\mathscr{B}_2$.

2. *The partial feedback behavior is called* well-posed *if it is an input/output behavior with input $\binom{u_1}{u_2}$ and output* $\binom{y_1}{y_2}=\begin{pmatrix}y_{11}\\ y_{12}\\ y_2\end{pmatrix}$. *This signifies that* $\begin{pmatrix}P_1 & -Q_{12}\\ -Q_2 & P_2\end{pmatrix}$ *is non-singular or has full rank* $p+m=\operatorname{rank}(P_1,-Q_{12})+\operatorname{rank}(-Q_2,P_2)$ *or, equivalently, that*

$$\mathscr{D}^{1\times p}(P_1,-Q_{12})\oplus\mathscr{D}^{1\times m}(-Q_2,P_2)=\mathscr{D}^{1\times(p+m)}\begin{pmatrix}P_1 & -Q_{12}\\ -Q_2 & P_2\end{pmatrix}.$$

3. *The plant* $\mathscr{B}_1$ *is* T-stabilizable by partial feedback *if there exists a compensator* $\mathscr{B}_2$ *such that* $\mathrm{fb}_{\mathrm{p}}(\mathscr{B}_1, \mathscr{B}_2)$ *is well-posed and* T*-stable. In this case any such compensator* $\mathscr{B}_2$ *is called a* T-stabilizing compensator *of* $\mathscr{B}_1$.

REMARK 2.2. *Note that* T*-stabilizability by* partial *feedback as defined above is a stronger property than the usual* T*-stabilizability which can be achieved by* full *feedback, i.e., where all inputs and outputs of the plant are available for the feedback interconnection. In the following a* T-stabilizing compensator *means a compensator that stabilizes the plant by* partial *feedback.*

LEMMA 2.3. *The partial feedback behavior* $\mathrm{fb}_{\mathrm{p}}(\mathscr{B}_1, \mathscr{B}_2)$ *is well-posed and* T*-stable if and only if* $\begin{pmatrix} P_1 & -Q_{12} \\ -Q_2 & P_2 \end{pmatrix} \in \mathrm{Gl}_{p+m}(\mathscr{D}_T)$. *This is equivalent to the direct sum decomposition* $\mathscr{D}_T^{1\times p}(P_1, -Q_{12}) \oplus \mathscr{D}_T^{1\times m}(-Q_2, P_2) = \mathscr{D}_T^{1\times(p+m)}$.

THEOREM 2.4 (Characterization of T-stabilizability by partial feedback). *The plant* $\mathscr{B}_1$ *is* T*-stabilizable by partial feedback if and only if the following two conditions are satisfied:*

1. *The matrix* $(P_1, -Q_{12})$ *has a right inverse in* $\mathscr{D}_T^{(p+m)\times p}$, *i.e.,* $\mathscr{B}_1' := \{\binom{y_1}{v_1} \in \mathscr{F}^{p+m};\ P_1 \circ y_1 = Q_{12} \circ v_1\}$ *is* T*-stabilizable (by full feedback) or the localization* $\mathscr{B}'_{1,T}$ *is controllable (see e.g. [2, Thm.3.12.2]).*
2. *The matrix* P_{11} *has a left inverse in* $\mathscr{D}_T^{p_1\times p}$, *i.e.,* y_{11} *is* T*-observable from* y_{12} *in* $\mathscr{B}_1^0$.

REMARK 2.5. *If the conditions of Theorem 2.4 are satisfied a constructive parametrization of all compensators* $\mathscr{B}_2$ *such that* $\mathrm{fb}(\mathscr{B}_1,\mathscr{B}_2)$ *is* T*-stable can be achieved. Also those compensators* $\mathscr{B}_2$ *such that* $\mathrm{fb}_{\mathrm{p}}(\mathscr{B}_1,\mathscr{B}_2)$ *is additionally proper can be constructed based on this parametrization. This procedure does however* not *yield a good characterization or construction of* proper *compensators* $\mathscr{B}_2$. *We choose another approach in the following: Under a slightly stronger condition than the one in Theorem 2.4 we construct a large class of* T*-stabilizing compensators* $\mathscr{B}_2$ *such that both* $\mathrm{fb}_{\mathrm{p}}(\mathscr{B}_1,\mathscr{B}_2)$ *and* $\mathscr{B}_2$ *are proper.*

ASSUMPTION 2.6. *Assume that* $(P_1, -Q_{12}) \in \mathscr{D}^{p\times(p+m)}$ *is right invertible over* $\mathscr{D}_T$. *Let* $(\widehat{P}_1, -\widehat{Q}_{12}) \in \widehat{\mathscr{D}}^{p\times(p+m)}$ *be the controllable realization of* $H_{12} := P_1^{-1}Q_{12}$ *over* $\widehat{\mathscr{D}}$, *i.e.,* $H_{12} = \widehat{P}_1^{-1}\widehat{Q}_{12}$ *a left coprime factorization of* H_{12} *over* $\widehat{\mathscr{D}}$ *[2, Lem.2.4]. Then* $(\widehat{P}_1, -\widehat{Q}_{12})$ *is right invertible over* $\widehat{\mathscr{D}}$ *and* $\mathscr{D}_T^{1\times p}(P_1, -Q_{12}) = \mathscr{D}_T^{1\times p}(\widehat{P}_1, -\widehat{Q}_{12})$ *since both* $(P_1, -Q_{12})$ *and* $(\widehat{P}_1, -\widehat{Q}_{12})$ *are controllable realizations of* H_{12} *over* $\mathscr{D}_T$.

ASSUMPTION 2.7. *Under Assumption 2.6 assume left invertibility of the submatrix* $\widehat{P}_{11}$ *of* $\widehat{P}_1 = (\widehat{P}_{11}, \widehat{P}_{12}) \in \widehat{\mathscr{D}}^{p\times(p_1+p_{12})}$ *over* $\mathscr{S}$. *Note that this is a slightly stronger assumption than the necessary condition according to Theorem 2.4 where left invertibility of* $\widehat{P}_{11}$ *– or equivalently of* P_{11} *– over* $\mathscr{D}_T$ *is required.*

ASSUMPTION 2.8. *Under Assumption 2.7 let* $\widehat{Q}_{11} \in \mathscr{D}_T^{p\times m_1}$ *be the (uniquely determined) matrix such that* $\mathscr{D}_T^{1\times p}(P_1, -Q_{12}, -Q_{11}) = \mathscr{D}_T^{1\times p}(\widehat{P}_1, -\widehat{Q}_{12}, -\widehat{Q}_{11})$. *Assume that* $\widehat{Q}_{11}$ *is contained in* $\mathscr{S}^{p\times m_1}$.

LEMMA 2.9. *Under Assumption 2.7 matrices* $\begin{pmatrix} \widehat{D}_2^0 \\ \widehat{N}_2^0 \end{pmatrix} \in \mathscr{S}^{(p+m)\times p}$ *and* $(-\widehat{Q}_2^0, \widehat{P}_2^0) \in \mathscr{S}^{m\times(p+m)}$ *where the submatrix* $\widehat{Q}_{20}^0 \in \mathscr{S}^{m\times p_1}$ *of* $\widehat{Q}_2^0 = (\widehat{Q}_{20}^0, \widehat{Q}_{21}^0) \in \mathscr{S}^{m\times(p_1+p_{12})}$ *is zero can be computed such that* $\begin{pmatrix} \widehat{P}_1 & -\widehat{Q}_{12} \\ -\widehat{Q}_2^0 & \widehat{P}_2^0 \end{pmatrix}\begin{pmatrix} \widehat{D}_2^0 & \widehat{N}_1 \\ \widehat{N}_2^0 & \widehat{D}_1 \end{pmatrix} = \mathrm{id}_{p+m}$ *and the following sequences are split exact:*

$$0 \longrightarrow \mathscr{S}^{1\times p} \xrightarrow{\circ(\widehat{P}_1, -\widehat{Q}_{12})} \mathscr{S}^{1\times(p+m)} \xrightarrow{\circ\begin{pmatrix} \widehat{N}_1 \\ \widehat{D}_1 \end{pmatrix}} \mathscr{S}^{1\times m} \longrightarrow 0,$$

$$0 \longleftarrow \mathscr{S}^{1\times p} \xleftarrow{\circ\begin{pmatrix} \widehat{D}_2^0 \\ \widehat{N}_2^0 \end{pmatrix}} \mathscr{S}^{1\times(p+m)} \xleftarrow{\circ(-\widehat{Q}_2^0, \widehat{P}_2^0)} \mathscr{S}^{1\times m} \longleftarrow 0.$$

In order to treat properness of the compensator $\mathscr{B}_2$ we use the equality $F(s)_{\rm pr} = \widehat{\mathscr{D}}_{\widehat{\mathscr{D}}\setminus\widehat{\mathscr{D}}\sigma}$ from [2, Lem.3.18] and the ring homomorphism

$$\nu_\sigma : F(s)_{\rm pr} \longrightarrow F, \quad r = \widehat{f}\widehat{g}^{-1} \longmapsto \widehat{f}(0)\widehat{g}(0)^{-1} \quad \text{where} \quad \widehat{f} \in \widehat{\mathscr{D}} = F[\sigma],\ \widehat{g} \in \widehat{\mathscr{D}} \setminus \widehat{\mathscr{D}}\sigma$$

from [2, Lem.3.17]. For matrices $M \in F(s)_{\rm pr}^{k\times\ell}$ we write $\nu_\sigma(M)$ for $(\nu_\sigma(M_{ij}))_{i,j}$.

THEOREM 2.10 (Construction of compensators). *Let $\mathscr{B}_1$ be a plant satisfying Assumption 2.8. With the notations from above and a universal left annihilator $\widehat{W} \in \widehat{\mathscr{D}}^{p_{12}\times p}$ of $\widehat{P}_{11}$ over $\widehat{\mathscr{D}}$ the following construction yields T-stabilizing compensators $\mathscr{B}_2$ such that both $\mathrm{fb_p}(\mathscr{B}_1,\mathscr{B}_2)$ and $\mathscr{B}_2$ are proper.*

- *Choose $Y_0 \in F^{m\times p_{12}}$ such that $\det\left(\nu_\sigma(\widehat{P}_2^0) - Y_0\widehat{W}(0)\widehat{Q}_{12}(0)\right) \neq 0$. Almost all $Y_0 \in F^{m\times p_{12}}$ satisfy this condition. With $\widehat{P}_2 := \widehat{P}_2^0 - Y\widehat{W}\widehat{Q}_{12}$ where $Y \in \mathscr{S}^{m\times p_{12}}$ is such that $\nu_\sigma(Y) = Y_0$ this signifies that $\nu_\sigma(\widehat{P}_2) \in \mathrm{Gl}_m(F)$ or that $\widehat{P}_2$ is non-singular and invertible over $F(s)_{\rm pr}$ by [2, Cor.3.23].*
- *Choose $Y_1 \in \mathscr{S}^{m\times p_{12}}$ arbitrarily and define $Y := Y_0 + \sigma Y_1$ and*

$$(-\widehat{Q}_2, \widehat{P}_2) := (-\widehat{Q}_2^0, \widehat{P}_2^0) + Y\widehat{W}(\widehat{P}_1, -\widehat{Q}_{12}) \in \mathscr{S}^{m\times(p+m)},$$
$$\begin{pmatrix}\widehat{D}_2\\ \widehat{N}_2\end{pmatrix} := \begin{pmatrix}\widehat{D}_2^0\\ \widehat{N}_2^0\end{pmatrix} - \begin{pmatrix}\widehat{N}_1\\ \widehat{D}_1\end{pmatrix} Y\widehat{W} \in \mathscr{S}^{(p+m)\times p}.$$

Then $\widehat{P}_2 \in \mathscr{S}^{m\times m} \cap \mathrm{Gl}_m(F(s)_{\rm pr})$ and $\widehat{Q}_{20} = 0$ where $\widehat{Q}_2 =: (\widehat{Q}_{20}, \widehat{Q}_{21}) \in \mathscr{S}^{m\times(p_1+p_{12})}$.
- *Choose $\widehat{Q}_{22} \in \mathscr{S}^{m\times p_2}$ arbitrarily and define $H_2 := (H_{22}, H_{21}) := \widehat{P}_2^{-1}(\widehat{Q}_{22}, \widehat{Q}_{21})$.*
- *Let $H_2 = (P_2^c)^{-1}(Q_{22}^c, Q_{21}^c)$ be a left coprime factorization of H_2 over $\mathscr{D}$ and choose $A \in \mathscr{D}^{m\times m} \cap \mathrm{Gl}_m(\mathscr{D}_T)$. Let $(-Q_{22}, -Q_{21}, P_2) := A(-Q_{22}^c, -Q_{21}^c, P_2^c)$ and $\mathscr{B}_2 := \left\{\begin{pmatrix}u_2\\ y_{12}\\ y_2\end{pmatrix} \in \mathscr{F}^{p_2+p_{12}+m};\ P_2 \circ y_2 = (Q_{22}, Q_{21}) \circ \begin{pmatrix}u_2\\ y_{12}\end{pmatrix}\right\}$.*

LEMMA 2.11. *In the situation of Assumption 2.6 $\widehat{P}_1^{-1}$ is proper iff $H_{12} := P_1^{-1}Q_{12}$ is proper. In this case the matrix $\widehat{Q}_{11}$ from Assumption 2.8 is proper iff $H_{11} := P_1^{-1}Q_{11}$ is so.*

We deduce that in particular the (necessary) conditions of Theorem 2.4 and properness of the transfer matrix $H_1 = (H_{12}, H_{11}) = P_1^{-1}(Q_{12}, Q_{11})$ of the plant $\mathscr{B}_1$ imply the conditions of Assumption 2.8 and are hence sufficient for applicability of Theorem 2.10. Assumption 2.8 may however be satisfied even if the plant $\mathscr{B}_1$ is not *proper.*

REMARK 2.12 (Pole placement). *If $\alpha \in F$ is chosen and Assumption 2.8 is satisfied for some saturated monoid $T \subseteq \mathscr{D} \setminus \{0\}$ containing $(s-\alpha)$ (for example for $T = \mathscr{D} \setminus \{0\}$), then there is a smallest possible saturated monoid T_1 containing $(s-\alpha)$ such that Assumption 2.8 is still satisfied for $T = T_1$ and $\mathscr{S} = \mathscr{S}_1 := \mathscr{D}_{T_1} \cap F(s)_{\rm pr}$. The set T_1 is the saturated monoid generated by the product t_1 of $(s-\alpha)$ and the greatest elementary divisors of $(P_1, -Q_{12})$ and of P_{11} over $\mathscr{D}$. Then the construction in Theorem 2.10 with $T = T_1$ and $\mathscr{S} = \mathscr{S}_1$ yields T_1-stabilizing compensators $\mathscr{B}_2$ of $\mathscr{B}_1$ such that both $\mathrm{fb_p}(\mathscr{B}_1,\mathscr{B}_2)$ and $\mathscr{B}_2$ are proper. Moreover, given any other saturated monoid $\widetilde{T}$ containing $(s-\alpha)$, $\mathscr{B}_1$ admits a $\widetilde{T}$-stabilizing compensator (such that $\mathrm{fb_p}(\mathscr{B}_1,\mathscr{B}_2)$ and $\mathscr{B}_2$ are proper) if and only if $T_1 \subseteq \widetilde{T}$. In particular, for any polynomial t_2 that is a multiple of $(s-\alpha)$ and the saturated monoid T_2 generated by t_2, there exists a T_2-stabilizing compensator $\mathscr{B}_2$ of $\mathscr{B}_1$ (such that $\mathrm{fb_p}(\mathscr{B}_1,\mathscr{B}_2)$ and $\mathscr{B}_2$ are proper) if and only if t_2 is a multiple of t_1.*

For the details of these considerations compare [2, Thm.3.15, Thm.4.11].

Finally we study the regulation problem. We consider again a plant $\mathscr{B}_1$ and are interested in constructing compensators $\mathscr{B}_2$ (by partial output feedback) that perform additional control tasks with the data from (2.1). We assume external behaviors

$$\mathscr{E}_1 = \{u_1 \in \mathscr{F}^{m_1}; V_1 \circ u_1 = 0\}, \quad \mathscr{E}_2 = \{u_2 \in \mathscr{F}^{p_2}; V_2 \circ u_2 = 0\},$$

$V_1 \in \mathscr{D}^{\ell_1 \times m_1}$, $V_2 \in \mathscr{D}^{\ell_2 \times p_2}$, and a linear operator $K = (K_y, K_u) \in \mathscr{D}^{e \times ((p+m)+(m_1+p_2))}$.

DEFINITION 2.13. *The T-stabilizing compensator $\mathscr{B}_2$ of the plant $\mathscr{B}_1$ is called a T*-regulator *(of $\mathscr{B}_1$ w.r.t. $\mathscr{E}_1$, $\mathscr{E}_2$, and K) if $K \circ \begin{pmatrix} y_1 \\ y_2 \\ u_1 \\ u_2 \end{pmatrix}$ is T-small for all $\begin{pmatrix} y_1 \\ y_2 \\ u_1 \\ u_2 \end{pmatrix} \in \mathrm{fb_p}(\mathscr{B}_1, \mathscr{B}_2)$ with $u_1 \in \mathscr{E}_1$ and $u_2 \in \mathscr{E}_2$.*

This definition contains important control problems such as for example tracking (any of the components of $y_1 = \binom{y_{11}}{y_{12}}$ and y_2 shall "track" any input u_1 satisfying the conditions of $\mathscr{E}_1$ and/or u_2 satisfying the conditions of $\mathscr{E}_2$), disturbance rejection (y_1 and y_2 or some components thereof shall not be influenced by any disturbances $u_1 \in \mathscr{E}_1$ and/or $u_2 \in \mathscr{E}_2$), combinations thereof etc.

THEOREM 2.14 (Characterization of T-regulators). *Assume a T-stabilizing compensator $\mathscr{B}_2$ of $\mathscr{B}_1$ constructed according to Theorem 2.10 and the data from above. Then $\mathscr{B}_2$ is a T-regulator if and only if there exists $Z \in \mathscr{D}_T^{e \times (\ell_1 + \ell_2)}$ such that*

$$K_y \begin{pmatrix} \widehat{D}_2 \widehat{Q}_{11} & \widehat{N}_1 \widehat{Q}_{22} \\ \widehat{N}_2 \widehat{Q}_{11} & \widehat{D}_1 \widehat{Q}_{22} \end{pmatrix} + K_u = Z \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}.$$

COROLLARY 2.15 (Existence of T-regulators). *Assume a plant $\mathscr{B}_1$ satisfying the conditions of Assumption 2.8, external behaviors $\mathscr{E}_1$ and $\mathscr{E}_2$, and a linear operator K as above. Then $\mathscr{B}_1$ admits a T-regulator $\mathscr{B}_2$ w.r.t. $\mathscr{E}_1$, $\mathscr{E}_2$, and K such that $\mathscr{B}_2$ and $\mathrm{fb_p}(\mathscr{B}_1, \mathscr{B}_2)$ are proper if there exist matrices $Y \in \mathscr{S}^{m \times p_{12}}$, $\widehat{Q}_{22} \in \mathscr{S}^{m \times p_2}$, and $Z \in \mathscr{D}_T^{e \times (\ell_1 + \ell_2)}$ such that*

$$K_u + K_y \begin{pmatrix} \widehat{D}_2^0 \widehat{Q}_{11} & 0 \\ \widehat{N}_2^0 \widehat{Q}_{11} & 0 \end{pmatrix} + K_y \begin{pmatrix} -\widehat{N}_1 Y \widehat{W} \widehat{Q}_{11} & \widehat{N}_1 \widehat{Q}_{22} \\ -\widehat{D}_1 Y \widehat{W} \widehat{Q}_{11} & \widehat{D}_1 \widehat{Q}_{22} \end{pmatrix} = Z \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \quad \textit{and} \tag{2.2}$$

$$\det\Big(\nu_\sigma(\widehat{P}_2^0) - \nu_\sigma(Y) \widehat{W}(0) \widehat{Q}_{12}(0)\Big) \neq 0. \tag{2.3}$$

REMARK 2.16. *The existence of $Y \in \mathscr{S}^{m \times p_{12}}$, $\widehat{Q}_{22} \in \mathscr{S}^{m \times p_2}$, and $Z \in \mathscr{D}_T^{e \times (\ell_1 + \ell_2)}$ such that the equation* (2.2) *and the inequality* (2.3) *are satisfied can be checked and all possible solutions can be constructed as described in [2] after Remark 4.7.*

COROLLARY 2.17 (Construction of T-regulators). *Assume that the conditions of Corollary 2.15 are satisfied. Then the following construction yields T-regulators $\mathscr{B}_2$ of $\mathscr{B}_1$ w.r.t. $\mathscr{E}_1$, $\mathscr{E}_2$, and K such that both $\mathrm{fb_p}(\mathscr{B}_1, \mathscr{B}_2)$ and $\mathscr{B}_2$ are proper.*

- *Choose $Y \in \mathscr{S}^{m \times p_{12}}$ and $\widehat{Q}_{22} \in \mathscr{S}^{m \times p_2}$ satisfying* (2.2) *and* (2.3) *for some $Z \in \mathscr{D}_T^{e \times (\ell_1 + \ell_2)}$, compare Remark 2.16. Define $(-\widehat{Q}_2, \widehat{P}_2) := (-\widehat{Q}_2^0, \widehat{P}_2^0) + Y \widehat{W} (\widehat{P}_1, -\widehat{Q}_{12})$. Then $\widehat{Q}_2 = (0, \widehat{Q}_{21}) \in \mathscr{S}^{m \times (p_1 + p_{12})} = \mathscr{S}^{m \times p}$ and $\widehat{P}_2 \in \mathscr{S}^{m \times m} \cap \mathrm{Gl}_m(F(s)_{\mathrm{pr}})$.*
- *Let $R_2^c := (-Q_{22}^c, -Q_{21}^c, P_2^c)$ be the controllable realization of $H_2 := \widehat{P}_2^{-1}(\widehat{Q}_{22}, \widehat{Q}_{21})$ over $\mathscr{D}$, i.e., $H_2 = (P_2^c)^{-1}(Q_{22}^c, Q_{21}^c)$ a left coprime factorization of H_2 over $\mathscr{D}$. Choose $A \in \mathscr{D}^{m \times m} \cap \mathrm{Gl}_m(\mathscr{D}_T)$ and define $(-Q_{22}, -Q_{21}, P_2) := A(-Q_{22}^c, -Q_{21}^c, P_2^c)$ and $\mathscr{B}_2 := \left\{ \begin{pmatrix} u_2 \\ y_{12} \\ y_2 \end{pmatrix} \in \mathscr{F}^{p_2 + p_{12} + m}; P_2 \circ y_2 = (Q_{22}, Q_{21}) \circ \binom{u_2}{y_{12}} \right\}$.*

REMARK 2.18 (Pole placement). *The pole placement problem can be treated similarly to Remark 2.12, compare also [2, Thm.4.11].*

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