ENTROPY FOR EXTERNAL STABILITY OF LINEAR CONTROL SYSTEMS*

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Abstract. For linear control systems, an entropy notion is introduced. It is associated with exponential stability of the output for every disturbance generated by an exosystem. This is motivated by problems with digital communication constraints from the plant to the control.

Key words. topological entropy, exosystems, disturbances

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1. Introduction.

This paper is concerned with minimal data rates for a problem of external stability in linear control systems subject to disturbances generated by an exosystem. In the context of classical geometric control theory, treatments of similar problems are given in the monographs Wonham [13], Basile and Marro [1], and Trentelman, Stoorvogel, Hautus [11]. The present paper concentrates on problems, where the information on the states is available for the control only via a digital communication channel; see Nair, Fagnani, Zampieri and Evans [10] for a survey. In particular, quantization approaches have been proposed to model the information flow from the states to the controller. Nair, Evans, Mareels, and Moran [9] have introduced the notion of topological feedback entropy, which may be viewed as a generalization of the notion of topological entropy in the theory of dynamical systems as presented in Walters [12]. The present note is based on concepts from Colonius and Kawan [5, 6], where invariance entropy for controlled invariance has been studied, and from Colonius and Helmke [4], Colonius [2]. This is based on counting the exponential growth rate of the number of required open loop controls as time increases to infinity. In Section 2, the approach is explained. Section 3 shows equivalence to an entropy for the uncontrolled system and presents a characterization in terms of eigenvalues.

2. Problem formulation.

We consider a linear control system together with an exosystem generating disturbances acting on the system. Let the exosystem in $\mathcal{X}_1$ be given by

$$\dot{x}_1(t) = A_1 x_1(t),$$

while the plant on $\mathcal{X}_2$ and its output are given by

$$\dot{x}_2(t) = A_3 x_1(t) + A_2 x_2(t) + B_2 u(t), z(t) = C_2 x_2(t).$$

Here $\mathcal{X}_1, \mathcal{X}_2$ and $\mathcal{Z}$ are finite dimensional normed spaces and $A_1 : \mathcal{X}_1 \to \mathcal{X}_1, A_2 : \mathcal{X}_2 \to \mathcal{X}_2, A_3 : \mathcal{X}_1 \to \mathcal{X}_2, B_2 : \mathbb{R}^m \to \mathcal{X}_2, C_2 : \mathcal{X}_1 \to \mathcal{Z}$ are linear. For definiteness, let $\dim \mathcal{X}_1 = n_1$ and $\dim \mathcal{X}_2 = n_2$ and for simplicity, identify $\mathcal{X}_1$ and $\mathcal{X}_2$ with the subspaces $\mathcal{X}_1 \times \{0\}$ and $\{0\} \times \mathcal{X}_2$, respectively, of $\mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2$. It is convenient to rewrite this as a system in $\mathcal{X}$ with

$$A := \begin{bmatrix} A_1 & 0 \\ A_3 & A_2 \end{bmatrix},\ B := \begin{bmatrix} 0 \\ B_2 \end{bmatrix},\ C := [0, C_2],$$

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so that we get

$$\dot{x}(t) = Ax(t) + Bu(t),\ z(t) = Cx(t) \tag{2.1}$$

with trajectories $\varphi(t, x_1, x_2, u) = (\varphi_1(t, x_1), \varphi_2(t, x_1, x_2, u)), t \geq 0$. We are interested in the problem to decouple the output from the disturbances, i.e., to make the output $z(t)$ independent of the disturbances generated by the exosystem. Suppose $u(t) \equiv 0$. Then, due to linearity, this amounts to the requirement, that for all initial values $x_1^0 \in \mathcal{X}_1$

$$C_2 \int_0^t e^{(t-s)A_2} A_3 e^{sA_1} x_1^0 ds = 0, t \geq 0.$$

Using state feedback $u = F_2 x_2$ one can replace A_2 by $A_2 + B_2 F_2$. However, if the present state (or measured output) is not available for control due to digital communication constraints, one may not be able to guarantee that the output vanishes identically. Instead, one has to relax this condition. We will require the following external stability property: the output $z(t)$ should be exponentially stable for every disturbance $x_1(t) = \varphi_1(t, x_1^0)$ determined by an initial value x_1^0. More precisely, let $\alpha > 0, M > 1$ and consider the following problem:

Find controls u such that for every $x_1^0 \in \mathcal{X}_1$ one has

$$\left\| Ce^{tA} x_1^0 + \int_0^t Ce^{(t-s)A} Bu(s) ds \right\| \leq Me^{-\alpha t} \left\| x_0^1 \right\|, t \geq 0; \tag{2.2}$$

Note that the left hand side equals

$$\left\| C_2 \varphi_2(t, x_1^0, u) \right\| = \left\| \int_0^t C_2 e^{A_2(t-s)} \left[A_3 e^{sA_1} x_1^0 + B_2 u(s) \right] ds \right\|.$$

Thus the plant is supposed to be initially in equilibrium, i.e., $x_2(0) = x_2^0 = 0$. The controls u may be generated by a compensator using measured outputs, see e.g. Trentelman, Stoorvogel, Hautus [11, Section 9.1] for a related discussion.

We will be concerned with problems, where the state or the observed outputs are not continually available for the controller and, assuming that the problem is solvable, we will discuss the exponential growth rate of the number of required control functions as time tend to infinity. This leads us to the following entropy notion.

Definition 2.1. *Fix $\alpha > 0$ and $M > 1$ and let $K_1 \subset \mathcal{X}_1$ be compact. For $T, \varepsilon > 0$ a set $\mathcal{R} \subset C([0,T], \mathbb{R}^m)$ of controls is (T, ε)-spanning if for all $y_1 \in K_1$ there is $u \in \mathcal{R}$ such that for all $t \in [0,T]$*

$$\left\| C_2 \varphi_2(t, y_1, u) \right\| < e^{-\alpha t} \left[\varepsilon + M \left\| y_1 \right\| \right]. \tag{2.3}$$

Let $r_{\text{ext}}(T, \varepsilon, K_1)$ be the minimal cardinality of such a (T, ε)-spanning set. The external stabilization entropy is defined by taking limits and the supremum over all compact $K_1 \subset \mathcal{X}_1$

$$h_{\text{ext}}(K_1, \alpha, M) := \lim_{\varepsilon \to 0} \overline{\lim_{T \to \infty}} \frac{1}{T} \log r_{\text{ext}}(T, \varepsilon, K_1), h_{\text{ext}}(\alpha, M) := \sup h_{\text{ext}}(K_1, \alpha, M).$$

Remark 2.2. *The number $r_{\text{ext}}(T, \varepsilon, K_1)$ depends also on α and M and, naturally, on the considered control system. Note that also h_{ext} depends on the flow $\Phi_t = e^{tA}$ and* *the output map C. Where convenient, we take the freedom to include those arguments which are relevant in the considered context. Furthermore, there may not exist finite (T,ε)-spanning sets (in particular, there may not exist any (T,ε)–spanning set). In this case, we set $r_{\text{ext}}(T,\varepsilon,K_1)=\infty$. Analogous conventions also apply to similar notions introduced below.*

We will relate the external stabilization entropy to a variant of topological entropy for the uncontrolled perturbed system $\dot{x}=Ax, z=Cx$. i.e.,

$$\begin{aligned}\dot{x}_1(t) &= A_1x_1(t) \\ \dot{x}_2(t) &= A_3x_1(t)+A_2x_2(t), z(t)=C_2x_2(t).\end{aligned} \tag{2.4}$$

This entropy notion refers to exponential convergence of the output and, again, it only refers to initial values in $\mathcal{X}_1$.

Definition 2.3. *Fix $\alpha>0$ and $M>1$ and let $K_1\subset\mathcal{X}_1$ be compact. For $T,\varepsilon>0$ a subset $R=\{x_1^1,...,x_1^r\}\subset K_1$ is (T,ε)-spanning if for every $y_1\in K_1$ there is $x_1^j\in R$ such that*

$$\left\|C_2\left[\varphi_2(t,y_1,0)-\varphi_2(t,x_1^j,0)\right]\right\|<e^{-\alpha t}\left(\varepsilon+M\left\|y_1\right\|\right) \text{ for all } t\in[0,T]. \tag{2.5}$$

Let $r_{\text{part}}(T,\varepsilon,K_1)$ be the minimal cardinality of such a (T,ε)-spanning set and define the partial exponential entropy by taking limits and the supremum over all compact $K_1\subset\mathcal{X}_1$

$$h_{\text{part}}(K_1,\alpha,M):=\lim_{\varepsilon\to 0}\overline{\lim_{T\to\infty}}\frac{1}{T}\log r_{\text{part}}(T,\varepsilon,K_1), h_{\text{part}}(\alpha,M)=\sup h_{\text{part}}(K_1,\alpha,M).$$

Remark 2.4. *We may assume that every minimal (T,ε)-spanning set R satisfies the following additional condition: for every $y_1\in K_1$ there is $x_1^j\in R$ such that $\left\|y_1-x_1^j\right\|<\varepsilon$. In fact, we can add finitely many initial points to a minimal (T,ε)-spanning set R such that this condition is satisfied. Their number does not depend on T. Hence the limit for $T\to\infty$ does not change, when we add these initial points.*

Remark 2.5. *We assume that for every $x_1^0\in\mathcal{X}_1$ there is a control u satisfying (2.2). Then, by compactness of K_1 and continuity with respect to y_1, there are finite (T,ε)-spanning sets for the external stabilization entropy. Analogous arguments can be applied to the partial exponential entropy.*

3. External stability. This section shows that the entropy notions introduced above are equivalent and presents estimates in terms of eigenvalues. First, the relation between these two notion of entropy is clarified by the following theorem which uses linearity in an essential way.

Theorem 3.1. *Fix $\alpha>0$ and $M>1$ and assume that for every $x_1^0\in\mathcal{X}_1$ there is a control u satisfying (2.2). Let $K_1\subset\mathcal{X}_1$ be compact. Then*

$$h_{\text{ext}}(K_1,\alpha,2M)\leq h_{\text{part}}(K_1,\alpha,M) \text{ and } h_{\text{part}}(K_1,\alpha,M)\leq h_{\text{ext}}(K_1,\alpha,2M).$$

Proof. (i) Fix $T,\varepsilon>0$. Consider a finite (T,ε)-spanning set $\mathcal{R}=\{u_1,...,u_r\}$ of controls with minimal cardinality $r=r_{\text{ext}}(T,\varepsilon,K_1)$ for the external stabilization entropy. Then for every $y_1\in K_1$ there is $u_j\in\mathcal{R}$ such that

$$\left\|C_2\varphi_2(t,y_1,u_j)\right\|<e^{-\alpha t}\left(\varepsilon+M\left\|y_1\right\|\right) \text{ for all } t\in[0,T].$$

By minimality, we can for every u_j pick $x_1^j \in K_1$ such that

$$\left\|C_2\varphi_2(t, x_1^j, u_j)\right\| < e^{-\alpha t}\left(\varepsilon + M\left\|x_1^j\right\|\right) \text{ for all } t \in [0,T].$$

Then one finds for all $y_1 \in K_1$ an element $x_1^j \in K_1$ such that for all $t \in [0,T]$

$$\begin{aligned}\left\|C_2\left[\varphi_2(t,y_1,0) - \varphi_2(t,x_1^j,0)\right]\right\| &= \left\|C_2\left[\varphi_2(t,y_1,u_j) - \varphi_2(t,x_1^j,u_j)\right]\right\| \\ &\le e^{-\alpha t}\left(\varepsilon + M\left\|y_1\right\|\right) + e^{-\alpha t}\left(\varepsilon + M\left\|x_1^j\right\|\right) \\ &\le e^{-\alpha t}\left(2\varepsilon + 2M\left\|y_1\right\| + M\left\|x_1^j - y_1\right\|\right) \\ &< e^{-\alpha t}\left(\varepsilon(2+M) + 2M\left\|y_1\right\|\right).\end{aligned}$$

Here we have also used Remark 2.4. This shows that $R := \{x_1^1, ..., x_1^r\}$ is a $(T, \varepsilon(2 + M))$-spanning set for the partial exponential entropy, and hence

$$r_{\text{part}}(T, \varepsilon(2+M), K_1, \alpha, 2M) \le r_{\text{ext}}(T, \varepsilon, K_1, \alpha, M).$$

Letting T tend to infinity and then ε tend to 0, one obtains

$$h_{\text{part}}(K_1, \alpha, 2M) \le h_{\text{ext}}(K_1, \alpha, M).$$

(ii) For the second inequality, let again $T, \varepsilon > 0$. Consider a minimal (T,ε)-spanning set $R = \{x_1^1, ..., x_1^r) \subset K_1$ for the partial exponential entropy which means that for all $y_1 \in K_1$ there is $j \in \{1, \ldots, r\}, r = r_{\text{part}}(T, \varepsilon, K_1, \alpha, M)$, such that for all $t \in [0,T]$

$$\left\|C_2\left[\varphi_2(t,y_1,0) - \varphi_2(t,x_1^j,0)\right]\right\| < e^{-\alpha t}\left(\varepsilon + M\left\|y_1\right\|\right) \text{ for all } t \in [0,T].$$

By assumption we can assign to each x_1^j a control u_j such that

$$\left\|C_2\varphi_2(t, x_1^j, u_j)\right\| < e^{-\alpha t}\left(\varepsilon + M\left\|x_1^j\right\|\right) \text{ for all } t \in [0,T].$$

Let $\mathcal{R} := \{u_1, \ldots, u_r\}$. Using linearity and Remark 2.4 we obtain that for every $y_1 \in K_1$ there is u_j such that for all $t \in [0,T]$

$$\begin{aligned}\left\|C_2\varphi_2(t,y_1,u_j)\right\| &\le \left\|C_2\left[\varphi_2(t,y_1,u_j) - \varphi_2(t,x_1^j,u_j)\right]\right\| + \left\|C_2\varphi_2(t,x_1^j,u_j)\right\| \\ &\le e^{-\alpha t}\left(\varepsilon + M\left\|y_1\right\|\right) + e^{-\alpha t}\left(\varepsilon + M\left\|x_1^j\right\|\right) \\ &\le e^{-\alpha t}\left(2\varepsilon + 2M\left\|y_1\right\| + M\left\|x_1^j - y_1\right\|\right) \\ &\le e^{-\alpha t}\left(\varepsilon(2+M) + 2M\left\|y_1\right\|\right).\end{aligned}$$

Hence $\mathcal{R}$ is $(T, \varepsilon(2+M), 2M)$-spanning for the external stabilization entropy and it follows that

$$r_{\text{ext}}(T, (2+M), K_1, \alpha, 2M) \le r_{\text{part}}(T, \varepsilon, K_1, \alpha, M) \text{ for all } T, \varepsilon > 0.$$

This yields $h_{\text{ext}}(K_1, \alpha, 2M) \le h_{\text{part}}(K_1, \alpha, M)$. □

Next we show that we may assume that all eigenvalues of A have positive real part. Decompose $\mathcal{X}$ into the invariant subspaces, $\mathcal{X} = \mathcal{X}^{-,\alpha} \oplus \mathcal{X}^{+,\alpha}$, where $\mathcal{X}^{-,\alpha}$

is the sum of all real generalized eigenspaces corresponding to eigenvalues with real part equal to or less than $-\alpha$ and $\mathcal{X}^{+,\alpha}$ is the sum of all real generalized eigenspaces corresponding to eigenvalues with real part greater than $-\alpha$. We denote the corresponding projections of $\mathcal{X}$ by $\pi^{-,\alpha}$ and $\pi^{+,\alpha}$, respectively, and let $\Phi^{-,\alpha}$ and $\Phi^{+,\alpha}$ be the associated restrictions of $\Phi_t = e^{At}$.

PROPOSITION 3.2. *Let $K_1 \subset \mathcal{X}_1$ be compact. Then the partial exponential entropies $h_{\text{part}}(K_1, \Phi)$ and $h_{\text{part}}(\pi^{+,\alpha}K_1, \Phi^{+,\alpha})$ (both with output map C) coincide.*

Proof. The restriction $\Phi^{-,\alpha}$ has the property, that for a polynomial $p(t)$

$$\left\| \Phi_t^{-,\alpha}(x-y) \right\| \leq p(t)e^{-\alpha t} \left\| x-y \right\|.$$

hence the partial exponential entropy here vanishes. Furthermore, the products of spanning sets for both parts yields spanning sets for the total system, hence

$$h_{\text{part}}(K_1, \Phi) \leq h_{\text{part}}(\pi^{+,\alpha}K_1, \Phi^{+}) + h_{\text{part}}(\pi^{-,\alpha}K_1, \Phi^{0,-}) = h_{\text{part}}(\pi^{+,\alpha}K_1, \Phi^{+}).$$

and clearly, $h_{\text{part}}(\pi^{+,\alpha}K_1, \Phi^{+}) \leq h_{\text{part}}(K_1, \Phi)$. □

The next proposition shows that only part of the state space $\mathcal{X}$ is relevant for the partial exponential entropy. Denote the smallest A_2-invariant subspace of $\mathcal{X}_2$ containing $\operatorname{Im} A_3$ by $\langle A_2 \,|\operatorname{Im} A_3 \rangle$ and denote the largest A_2-invariant subspace of $\operatorname{Im} A_3 \cap \ker C_2$ by $\ker(A_2; \operatorname{Im} A_3 \cap \ker C_2)$. Let the natural projection be π,

$$\pi : \langle A_2 \,|\operatorname{Im} A_3 \rangle \to \langle A_2 \,|\operatorname{Im} A_3 \rangle \,/\, \ker(A_2; \operatorname{Im} A_3 \cap \ker C_2) =: \bar{\mathcal{X}}_2.$$

Then π induces on $\mathcal{X}_1 \times \bar{\mathcal{X}}_2$ the system

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) \\ \frac{d}{dt}\overline{x}_2(t) &= \bar{A}_2 \overline{x}_2(t) + \pi A_3 x_1(t), z(t) = \overline{C}_2 \overline{x}_2(t). \end{aligned} \tag{3.1}$$

PROPOSITION 3.3. *For every compact $K_1 \subset \mathcal{X}_1$ the partial exponential entropies of system (2.2) on $\mathcal{X}_1 \times \mathcal{X}_2$ and of the induced system given by (3.1) on $\mathcal{X}_1 \times \bar{\mathcal{X}}_2$ coincide,*

$$h_{\text{part}}(K_1) = \overline{h}_{\text{part}}(K_1).$$

In view of this proposition we will in the following consider the induced flow on $\mathcal{X}_1 \times \bar{\mathcal{X}}_2$. We recall some facts on topological entropy of linear flows. For a linear map $A : \mathcal{X} \to \mathcal{X}$ on an n-dimensional normed vector space $\mathcal{X}$, let $\Phi_t := e^{tA}$ and let $K \subset \mathcal{X}$ be compact. Here a set $R \subset K$ in $\mathcal{X}$ is called (T, ε)-spanning if for every $x \in K$ there is $y \in R$ such that for all $t \in [0, T]$ one has

$$\left\| \Phi_t x - \Phi_t y \right\| = \left\| e^{tA}(x-y) \right\| < \varepsilon.$$

Denote the minimal cardinality of such a (T, ε, K)-spanning set by $r_{\text{top}}(T, \varepsilon, K)$.

DEFINITION 3.4. *With the notation above, the topological entropy of Φ with respect to compact sets K in a subspace V is defined by*

$$h_{\text{top}}(K) := \lim_{\varepsilon \searrow 0} \overline{\lim_{T\to\infty}} \frac{1}{T} \log r_{\text{top}}(T, \varepsilon, K), h_{\text{top}}(V) := \sup_{K \subset V} h_{\text{top}}(K).$$

We recall the following results from Colonius, San Martin, da Silva [8], see also Colonius [3].

THEOREM 3.5. *Consider a linear map* $A : \mathcal{X} \to \mathcal{X}, \dim \mathcal{X} = n$, *with flow* $\Phi_t = e^{tA}$ *and assume that* Φ *is hyperbolic. Let* V *be a* k*-dimensional subspace. Then the topological entropy of* V *is*

$$h_{\text{top}}(V, \Phi) = \sum_{i=1}^{l} k_i \max(0, \lambda_i).$$

The expression on the right hand side is determined in the following way: Let $L_i, i = 1, \ldots, l$, *be the Lyapunov spaces of* A *corresponding to the Lyapunov exponents* λ_i. *For* $k \in \{1, \ldots, n\}$ *define the index set*

$$I(k) = \{(k_1, \ldots, k_l) \mid k_1 + \ldots + k_l = k \text{ and } 0 \le k_i \le \dim L_i\}. \tag{3.2}$$

Then there is a unique l*-tuple* $(k_1, ..., k_l) \in I(k)$ *such that the omega-limit set* $\omega(V)$ *of the subspace* V *in the Grassmannian* $\mathbb{G}_k\mathcal{X}$ *is contained in*

$$\mathbb{G}_{k_1} L_1 \oplus \ldots \oplus \mathbb{G}_{k_l} L_l, \ (k_1, \ldots, k_l) \in I(k), \tag{3.3}$$

which denotes the set of all k*-dimensional subspaces* $V^k \subset \mathcal{X}$ *with* $\dim(V^k \cap L_i) = k_i, \ i = 1, \ldots, l$.

Returning to the partial entropy, consider the induced flow $\bar{\Phi}_t = e^{t\bar{A}}$ on $\mathcal{X}_1 \times \bar{\mathcal{X}}_2$ with

$$\bar{\mathcal{X}}_2 := \langle A_2 \,|\text{Im}\, A_3 \rangle / \ker(A_2; \text{Im}\, A_3 \cap \ker C_2).$$

We will show that the partial exponential entropy is bounded above by the topological entropy of $\mathcal{X}_1$ for the induced flow $\bar{\Phi}_t^\alpha = e^{t(\bar{A}+\alpha I)}$.

THEOREM 3.6. *The partial exponential entropy of (2.1) and the topological entropy of* $\mathcal{X}_1$ *for the induced system on* $\mathcal{X}_1 \times \bar{\mathcal{X}}_2$ *satisfy*

$$h_{\text{part}}(\alpha, M) \le h_{\text{top}}(\mathcal{X}_1, \bar{\Phi}^\alpha) = \sum k_i(\alpha + \lambda_i), \tag{3.4}$$

where summation is over the real parts $\lambda_i > -\alpha$ *of the eigenvalues of* $\bar{A}$ *and the* k_i *are given by the the set of the form (3.3) in the* n_1*-Grassmannian* $\mathbb{G}_{n_1}(\mathcal{X}_1 \times \bar{\mathcal{X}}_2)$ *containing the* ω*-limit set* $\omega(\mathcal{X}_1)$ *with respect to the induced flow* $\bar{\Phi}_t$.

Proof. Let $T, \varepsilon > 0$ and consider a (T, ε)-spanning set $R = \{x_1^1, ..., x_1^r\} \subset K_1$ with minimal cardinality $r = r_{\text{top}}(T, \varepsilon, K_1)$ for the topological entropy of the flow $\bar{\Phi}_t^\alpha = e^{t(\bar{A}+\alpha I)}$. For every $y_1 \in K_1$ there exists $x_1^j \in R$ such that for all $t \in [0, T]$

$$e^{\alpha t} \left\| e^{t\bar{A}}(y_1 - x_1^j) \right\| = \left\| e^{t(\bar{A}+\alpha I)}(y_1 - x_1^j) \right\| < \varepsilon,$$

and hence $\left\| e^{t\bar{A}}(y_1 - x_1^j) \right\| < e^{-\alpha t}\varepsilon$. In particular, the second component satisfies

$$\left\| \varphi_2(t, y_1 - x_1^j, 0) \right\| = \left\| \int_0^t e^{(t-\tau)\bar{A}_2} \pi A_3 \left[\varphi_1(\tau, y_1) - \varphi_1(t, x_1^j) \right] d\tau \right\| < e^{-\alpha t}\varepsilon.$$

Thus R is also (T, ε)-spanning for the partial exponential entropy. It follows that the minimal cardinality $r_{\text{top}}(T, \varepsilon, K_1, \bar{\Phi}^\alpha)$ for the topological entropy is greater than or equal to the minimal cardinality $r_{\text{part}}(T, \varepsilon, K_1)$ for the partial exponential entropy. Then take the limit superior for $T \to \infty$ and let ε tend to 0. Finally, the expression for the topological entropy follows from Theorem 3.5. □

In the following we will introduce conditions which ensure equality in (3.4). In view of Proposition 3.2 and Proposition 3.3 we can assume that all eigenvalues of A have real parts greater than $-\alpha$ and we can replace $\mathcal{X}_2$ by the quotient space $\bar{\mathcal{X}}_2 = \langle A_2 \,|\mathrm{Im}\, A_3 \rangle / \ker(A_2; \mathrm{Im}\, A_3 \cap \ker C_2)$ which we again denote by $\mathcal{X}_2$.

We impose the following assumption on volumes: Let $\pi : \mathcal{X} \to \mathcal{X} / \ker C$ be the natural projection. Take an orthonormal basis $v_1, ..., v_{n_1}$ of $\mathcal{X}_1$ (we may assume that $\mathcal{X}$ is endowed with an inner product.) There is $\gamma > 0$ such that for a sequence $t_i \to \infty$ the absolute value of the volume of the parallelepiped spanned by $\pi(e^{t_i A} v_1), ..., \pi(e^{t_i A} v_{n_1}) \subset \mathcal{X} / \ker C$ is bounded below by a positive constant times the absolute value of the volume of the parallelepiped spanned by $e^{t_i A} v_1, ..., e^{t_i A} v_{n_1}$. More formally, we require:

There is an orthonormal basis $v_1, ..., v_{n_1}$ of $\mathcal{X}_1$ such that some $\gamma > 0$ and a sequence $t_i \to \infty$

$$\left\| \pi(e^{t_i A} v_1) \wedge \cdots \wedge \pi\left(e^{t_i A} v_{n_1}\right) \right\| \geq \gamma \left\| e^{t_i A} v_1 \wedge \cdots \wedge e^{t_i A} v_{n_1} \right\|. \tag{3.5}$$

Note that this assumption can only hold, if $\dim \mathcal{X} / \ker C = \dim \mathcal{X}_2 / \ker C_2 \geq \dim \mathcal{X}_1$.

PROPOSITION 3.7. *Assume that condition (3.5) holds. Then the partial exponential entropy of (2.1) is given by*

$$h_{\mathrm{part}}(\alpha, M) = h_{\mathrm{top}}(\mathcal{X}_1, \bar{\Phi}^{\alpha}).$$

Proof. A consequence of (3.5) is that

$$\limsup_{t\to\infty} \frac{1}{t} \log \left\| \pi(e^{tA} v_1) \wedge \cdots \wedge \pi\left(e^{tA} v_{n_1}\right) \right\| \geq \limsup_{t\to\infty} \frac{1}{t} \log \left\| e^{tA} v_1 \wedge \cdots \wedge e^{tA} v_{n_1} \right\|. \tag{3.6}$$

Let P be the parallelepiped spanned by $v_1, ..., v_{n_1}$. Then the set $e^{tA}P$ is the parallelepiped spanned by $e^{tA}v_1, \cdots, e^{tA}v_{n_1}$. The projected set $\pi(e^{tA}P)$ is the parallelepiped spanned by $\pi(e^{tA_k}v_1), \cdots, \pi\left(e^{tA_k}v_{n_1}\right)$. By Colonius and Kliemann [7, Theorem 5.2.5] one finds

$$\lim_{t\to\infty} \frac{1}{t} \log \left\| e^{tA} v_1 \wedge \cdots \wedge e^{tA} v_{n_1} \right\| = \sum_{i=1}^{l} k_i \lambda_i,$$

where $(k_1, ..., k_l)$ is an element of the index set $I(k)$ given by (3.2). In order to relate the volume growth to the partial exponential entropy, we argue similarly as in Colonius, San Martin, da Silva [8, Proposition 4.1]: For $t > 0$ the n_1-dimensional volume of $\pi(e^{tA}P)$ satisfies

$$\mathrm{vol}^k(\pi(e^{tA}K)) \geq \mathrm{vol}^k(\pi(e^{tA}P)) = \left\| \pi(e^{tA} v_1) \wedge \cdots \wedge \pi\left(e^{tA} v_{n_1}\right) \right\|.$$

Let $k_1 := \max_{i=1,...,n_1} \|v_i\|$, fix $\varepsilon > 0, T > 0$, and consider a (T, ε)-spanning set $R = \{x_1^1, ..., x_1^r\} \subset P$ of minimal cardinality $r = r_{\mathrm{part}}(T, \varepsilon, P)$ for the partial exponential entropy. Then we claim that the set $\pi(e^{TA}P)$ is contained in the union of r balls $B(\pi(e^{TA}x_j); e^{-\alpha t}(\varepsilon + k_1)$ of radius $e^{-\alpha t}(\varepsilon + k_1)$ in $\mathcal{X} / \ker C$,

$$B(\pi(e^{TA}x_j); e^{-\alpha t}(\varepsilon + k_1)) = \{z \in \mathcal{X} / \ker C \mid \left\| z - \pi(e^{TA}x_j) \right\| < e^{-\alpha t}(\varepsilon + k_1)\}.$$

In fact, by Definition 2.3, for every $y_1 \in P$ there is $x_1^j \in R$ such that

$$\left\| C_2 \left[\varphi_2(t, y_1, 0) - \varphi_2(t, x_1^j, 0) \right] \right\| < e^{-\alpha t} \left(\varepsilon + M \left\| y_1 \right\|\right) \text{ for all } t \in [0, T].$$

This is equivalent to

$$\left\| C_2 \left[\varphi_2(t, y_1 - x_1^j, 0) \right] \right\| = \mathrm{dist}(\varphi(t, y_1 - x_1^j, 0), \ker C) = \left\| \pi \left[e^{At}(y_1 - x_1^j) \right] \right\|,$$

where the last expression denotes the norm in the quotient space $\mathcal{X}/\ker C$. Hence for every $y_1 \in P$ there is $x_1^j \in R$ such that

$$\left\| \pi \left[e^{At}(y_1 - x_1^j) \right] \right\| < e^{-\alpha t} \left(\varepsilon + M \left\| y_1 \right\| \right) \text{ for all } t \in [0, T].$$

This shows the claim. Each such ball has volume bounded by $c\,(\varepsilon + Mk_1))^{n_2}$, where $c > 0$ is a constant. Thus

$$\mathrm{vol}^{n_1}(\pi(e^{TA}P)) \leq r \cdot c\,(\varepsilon + Mk_1))^{n_2}.$$

This yields

$$\begin{aligned} \log r = \log r_{\mathrm{part}}(T, \varepsilon, P) &\geq \log \mathrm{vol}^{n_1}(\pi(e^{TA}P)) - \log\left[c\,(\varepsilon + Mk_1))^{n_2}\right] \\ &= \log \left\| \pi(e^{TA}v_1) \wedge \cdots \wedge \pi\left(e^{TA}v_{n_1}\right) \right\| - \log\left[c\,(\varepsilon + Mk_1))^{n_2}\right], \end{aligned}$$

and hence

$$\limsup_{T\to\infty} \frac{1}{T} \log r_{\mathrm{part}}(T, \varepsilon, P) \geq \limsup_{T\to\infty} \frac{1}{T} \log \left\| \pi(e^{TA}v_1) \wedge \cdots \wedge \pi\left(e^{TA}v_{n_1}\right) \right\|.$$

Together with (3.6) one obtains the assertion for $\varepsilon \to 0$. □

4. Conclusions. This paper introduces a way to measure the minimal information that is needed in order to obtain external stability in linear control systems. Here the exponential growth rate of the number of open loop controls is measured which allow us to obtain exponential stability of the output for every disturbance generated by an exosystem. Theorem 3.6 provides an upper bound in terms of topological entropy of an induced differential equation. This bound can be expressed in terms of the real parts of certain eigenvalues of A.

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