LINEAR-EXPONENTIAL-QUADRATIC GAUSSIAN CONTROL FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper a control problem for a controlled linear stochastic equation in a Hilbert space and an exponential quadratic cost functional of the state and the control is formulated and solved. The stochastic equation can model a variety of stochastic partial differential equations with the control restricted to the boundary or to discrete points in the domain. The solution method does not require solving a Hamilton-Jacobi-Bellman equation and the method provides an explanation for an additional term in the Riccati equation as compared to the Riccati equation for a quadratic cost functional. The optimal cost is also given explicitly. Some examples of controlled stochastic partial differential equations are given.

Key words. linear exponential quadratic control, control of partial differential equations, control of stochastic equations in a Hilbert space

AMS subject classifications. 60H15, 60G18, 93H20

1. Introduction.

An important generalization of the linear-quadratic Gaussian (LQG) control problem is the linear-exponential-quadratic Gaussian (LEQG) control problem particularly for its application in risk sensitive control and its relation to differential games. An LEQG problem is similar to an LQG control problem except that the cost is an exponential of a quadratic functional of the state and the control instead a quadratic functional. The LEQG problem for finite dimensional linear systems is solved in [11] by determining a solution to the Hamilton-Jacobi-Bellman (HJB) equation associated with this stochastic control problem. A different approach to the solution of this finite dimensional problem is given in [3] where a combination of the methods of completion of squares and absolute continuity of measures is used for the solution. This latter approach provides an explanation for the additional term of the Riccati equation for the LEQG problem as compared with the Riccati equation for the LQG problem and this approach is more elementary and direct than solving the HJB equation for the LEQG problem.

A natural generalization of this LEQG control problem for systems in finite dimensions is to linear stochastic equations in an infinite dimensional Hilbert space that can model various types of controlled linear stochastic partial differential equations. In this paper such a problem is formulated and solved. A semigroup approach is used where the semigroups are analytic [15]. The control can be restricted to discrete points in the domain or to the boundary of the domain to describe a typical controlled physical system and is the primary reason for restriction to analytic semigroups. Thus in addition to the infinitesimal generator acting on the state, the linear transformation acting on the control is also an unbounded operator so that properties of the solution of the Riccati equation require more refinement than for distributed control to ensure that the optimal control in the system equation is well defined.

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2. Preliminaries. The controlled linear stochastic system is described by the following stochastic differential equation

$$\begin{aligned} dX(t) &= AX(t)dt + BU(t)dt + \Phi dW(t) \qquad (2.1)\\ X(0) &= X_0 \end{aligned}$$

where $X(t) \in H$ for $t \in [0,T]$, H is a real, separable, infinite dimensional Hilbert space, and $(W(t), t \in [0,T])$ is a standard cylindrical Wiener process in H. The complete probability space is denoted $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathbb{P}$ is induced from the standard cylindrical measure for the Wiener process and $\mathcal{F}$ is the $\mathbb{P}$-completion of the Borel σ-algebra on Ω. Let $(\mathcal{F}(t), t \in [0,T])$ be an increasing $\mathbb{P}$-complete family of sub-σ-algebras of $\mathcal{F}$ such that $X(t)$ is $\mathcal{F}(t)$ measurable for each $t \in [0,T]$ and $(< l, W(t) >, t \in [0,T])$ is a Brownian martingale with local variance $|l|_H^2$ for each nonzero $l \in H$. The linear operator A is the infinitesimal generator of an analytic semigroup on H (e.g. [15]). Thus for some $\beta > 0$ the operator $-A + \beta I$ is strictly positive so that the fractional powers $(-A + \beta I)^{\gamma}$ and $(-A^* + \beta I)^{\gamma}$ and the spaces $D_A^{\gamma} = D((-A + \beta I)^{\gamma})$ and $D_{A^*}^{\gamma} = D((-A^* + \beta I)^{\gamma})$ with the graph norm topology for $\gamma \in \mathbb{R}$ can be defined. The linear space $D(\cdot)$ denotes the domain of $\cdot$. It is assumed that $B \in \mathcal{L}(H_1, D_A^{\epsilon-1})$ where H_1 is a real, separable Hilbert space and $\epsilon \in (0,1)$. The linear operator Φ is assumed to be Hilbert-Schmidt. It is assumed that for each $x \in H$ there is a $u_x \in L^2([0,T], H_1)$ such that

$$y(\cdot) = S(\cdot)x + \int_0^{\cdot} S(\cdot - r)Bu_x(r)dr \in L^2([0,T], H) \qquad (2.2)$$

The cost functional J is an exponential of a quadratic functional of X and U that is given by

$$\begin{aligned} J(U) = \mathbb{E}\exp[\frac{\mu}{2}\int_0^T &< QX(s), X(s) > + < RU(s), U(s) > ds \\ &+ \frac{\mu}{2} < MX(T), X(T) >] \qquad (2.3) \end{aligned}$$

where $T > 0$ is fixed, $\mu > 0$ is fixed, and Q and R are strictly positive, self-adjoint operators.

The Riccati equation to solve the LQG problem with the linear stochastic system (2.1) and the quadratic cost that appears in the exponential function (2.3) is the following formal equation

$$-\frac{dP}{dt} = A^*P + PA - PBR^{-1}B^*P + Q \qquad (2.4)$$

$$P(T) = M \qquad (2.5)$$

The equation (2.4) can be modified to a mathematically meaningful inner product equation as

$$\begin{aligned} -\frac{d}{dt} < Px, y > = &< Ax, Py > + < Px, Ay > - < R^{-1}B^*Px, B^*y > \qquad (2.6)\\ &+ < Qx, y > \end{aligned}$$

for $x, y \in D(A)$. It is known that there is a unique, nonnegative self-adjoint solution of (2.6) (cf. [2], [8], [9], [13]).

The family of admissible controls, $\mathcal{U}$, is

$\mathcal{U} = \{U : [0,T] \times \Omega \to H_1 | \ U$ is adapted to $(\mathcal{F}(t), t \in [0,T])$ and $\int_0^T |U(t)|^p dt < \infty$ a.s.$\}$

where $p > max\{2, 1/\epsilon\}$ is fixed.

3. Main Result. In this section an optimal control is explicitly given for the control problem for the linear system (2.1) and the cost (2.3). The authors are not aware of any previous results for an optimal control for an exponential quadratic cost with a linear stochastic system with boundary or point control in a general Hilbert space.

THEOREM 3.1. *The optimal control problem given by* (2.1) *and* (2.3) *has an optimal control,* $(U^*(t), t \in [0,T])$*, in* $\mathcal{U}$ *that is given by*

$$U^*(t) = -R^{-1}C^T P(t)X(t) \tag{3.1}$$

where $(P(t), t \in [0,T])$ *is assumed to be the unique, symmetric, positive* $\mathcal{L}(H, D_{A^*}^{1-\epsilon})$*-valued solution of the following Riccati equation*

$$\begin{aligned} -\frac{d}{dt} < Px, y > &= < xA, Py > + < Px, Ay > - < R^{-1}B^*Px, B^*Py > \\ &\quad - \mu < \Phi^*Px, \Phi^*Py > + < Qx, y > \\ < P(T)x, y > &= < Mx, y > \end{aligned} \tag{3.2}$$

for $x, y \in D(A)$ *and the optimal cost is*

$$J(U^*) = G(0)exp[\frac{\mu}{2} < P(0)X_0, X_0 >] \tag{3.3}$$

and $(G(t), t \in [0,T])$ *satisfies*

$$\begin{aligned} -\frac{dG}{dt} &= \frac{\mu}{2} Gtr(P\Phi\Phi^*) \\ G(T) &= 1 \end{aligned} \tag{3.4}$$

Sketch of proof. Initially a completion of squares is made of the terms that appear in the exponent of the cost using the methods in [5], [6]. With the completion of the square there are three terms that do not occur in the square of an affine functional of the control. One of these terms determines the optimal cost and the other two terms provide a (local) Radon-Nikodym derivative that transforms the Wiener measure for $(\Phi W(t), t \in [0,T])$ by addition of a drift term. Thus all of the terms in the exponent are accounted and the optimal control and the optimal cost follow. A complete proof of this theorem is given in [7].

The difference between the Riccati equation (2.6) for the LQG problem and the Riccati equation (3.2) for the LEQG problem is the term $-\mu < \Phi^*Px, \Phi^*Py >$ that arises from the quadratic term in the exponential function for a Radon-Nikodym derivative that transforms the Wiener measure for $(\Phi W(t), t \in [0,T])$ by adding a drift term that appears as a stochastic integral. For the completion of squares for the LQG problem the stochastic integral term has expectation zero, so it disappears with the operation of expectation. For the completion of squares for the LEQG problem there is an exponential of the stochastic integral term so that it does not have expectation zero. The Radon-Nikodym derivative (exponential martingale) is the natural way to eliminate this exponential of a stochastic integral.

4. Some Examples. Some examples are given now that indicate the range of applicability of the optimal control result.

Example 1. This is a family of examples from elliptic differential operators which is discussed in more detail in [4]. Let G be a bounded, open domain in $\mathbb{R}^n$ with C^∞-boundary ∂G with G locally on one side of ∂G and let $L(x,D)$ be an elliptic differential operator of the form

$$L(x,D)f = \Sigma_{i,j=1}^n D_i a_{ij}(x) D_j f + \Sigma_{i=1}^n [b_i(x) D_i f + D_i(d_i(x)f)] + c(x)f \tag{4.1}$$

where the coefficients a_{ij}, b_i, d_i, c are elements of $C^\infty(G)$

$$\Sigma a_{ij}(x)\xi_i\xi_j \geq \hat{\nu}|\xi|^2 \tag{4.2}$$

where $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n, x \in G$, $\hat{\nu} > 0$ is a constant, and $\{a_{ij}\}$ is symmetric. Consider a stochastic parabolic control problem formally described by the equations

$$\frac{\partial y}{\partial t} = L(x,D)y(t) + \eta(t,x) \tag{4.3}$$

for $(t,x) \in \mathbb{R}_+ \times G$ and

$$\frac{\partial y}{\partial t} + h(x)y(t,x) = u(t,x) \tag{4.4}$$

for $(t,x) \in \mathbb{R}_+ \times \partial G$ and $y(0,x) = y_0(x)$ where $\frac{\partial}{\partial \nu} = \Sigma_{i,j=1}^n a_{ij}\nu_j D_i$ is the outward normal derivative, $\nu = (\nu_1, ..., \nu_n)$ is the unit outward normal to ∂G, the process $(\eta(t,x), (t,x) \in \mathbb{R}_+ \times G)$ formally denotes a space dependent white noise, $u \in L^2(0,T,L^2(\partial G))$, $h \in C^\infty(\partial G)$, and $h \geq 0$.

To give a mathematical description to (4.3) and (4.4), a semigroup approach [15] is used. Let $H = L^2(G), H_1 = L^2(\partial G)$ and define the infinitesimal generator as $Af = L(x,D)f$ so that $A : D(A) \to H$ and $D(A) = \{f \in H^2(G) : \frac{\partial f}{\partial \nu} = 0 \text{ on } \partial G\}$. It is well known that A generates an analytic semigroup (e.g. [15]) and the linear operator $(A - \beta I)$ is strictly negative for some $\beta \geq 0$.

To define the control operator in the stochastic equation, consider the elliptic problem

$$\begin{aligned} (L(x,D) - \beta)z &= 0 \qquad \text{on } G \qquad (4.5)\\ \frac{\partial z}{\partial \nu} + hz &= -g \qquad \text{on } \partial G \qquad (4.6) \end{aligned}$$

For $g \in L^2(\partial G)$, these is a unique solution $z \in H^{\frac{3}{2}}$ [14]. Define $\hat{B} \in \mathcal{L}(H_1, H^{\frac{3}{2}})$ by the equation, $\hat{B}g = -z$. For $\epsilon < \frac{3}{4}$, $\hat{B} \in \mathcal{L}(H_1, D_A^\epsilon)$ because $D_A^{\frac{3}{4}-\gamma} = H^{\frac{3}{2}-2\gamma}$ for a sufficiently small $\gamma > 0$ [10]. Let $y_\beta(t,x) = e^{-\beta t}y(t,x)$ and $\eta(t,x)dt = \Phi dW(t)$ for some $\Phi \in \mathcal{L}(H)$ and a standard cylindrical Wiener process $(W(t), t \in [0,T])$ in H. From (4.5), (4.6) it follows that

$$\begin{aligned} dy_\beta &= (L(x,D) - \beta)y_\beta dt + e^{-\beta t}\Phi dW(t) \qquad (4.7)\\ \frac{\partial y_\beta}{\partial \nu} + hy_\beta &= e^{-\beta t}u = u_\beta(t) \qquad \text{on } \partial G \qquad (4.8)\\ y_\beta(0) &= y(0) \qquad (4.9) \end{aligned}$$

Formally performing the differentiation $(\frac{\partial}{\partial t})\hat{B}u_\beta(t)$, it follows that

$$d\omega_\beta(t) = ((L(x,D) - \beta)y_\beta(t) - \hat{B}v_\beta(t))dt + e^{-\beta t}\Phi dW(t) \tag{4.10}$$

$$\frac{\partial \omega_\beta}{\partial \nu} + h\omega_\beta = 0 \qquad \text{on } \mathbb{R}_+ \times \partial G \tag{4.11}$$

where v_β is the formal time derivative of u_β and $\omega_\beta(t) = y_\beta(t) - \hat{B}u_\beta(t)$. For (4.7) the mild solution is

$$\begin{aligned}\omega_\beta(t) = S_\beta(t)(y(0) + \hat{B}u(0)) &+ \int_0^t S_\beta(t-r)\Phi e^{\beta r} dW(r) \\ &- \int_0^t S_\beta(t-r)\hat{B}v_\beta(r)dr\end{aligned} \tag{4.12}$$

where $S_\beta(t) = e^{t(A-\beta I)}$. Formally integrating by parts in the Lebesgue integral in (4.12) and canceling the term $e^{-\beta t}$ gives

$$y(t) = S(t)y(0) + \int_0^t S(t-r)Bu(r)dr + \int_0^t S(t-r)\Phi dW(r) \tag{4.13}$$

which is a mild solution to a stochastic equation of the form (2.1) where $B = \Psi^*$ and $\Psi^* \in \mathcal{L}(D_{A^*}^{1-\epsilon}, H_1)$ extends the linear operator $\hat{B}^*(A^* - \beta I)$.

Example 2. A second example is a structurally damped plate with random loading and point control (cf. [4] for more details). Consider the following model of a plate in the deflection ω

$$\omega_{tt}(t,x) + \Delta^2\omega(t,x) - \alpha\Delta\omega(t,x) = \delta(x - x_0)u(t) + \eta(t,x) \tag{4.14}$$

$$\text{for } (t,x) \in \mathbb{R}_+ \times G$$

$$\omega(0,\cdot) = \omega_0 \qquad \omega_t(0,\cdot) = \omega_1 \tag{4.15}$$

$$\omega|_{\mathbb{R}_+\times\partial G} = \Delta\omega|_{\mathbb{R}_+\times\partial G} = 0 \tag{4.16}$$

where $\alpha > 0$ is a constant, $\eta(t,x)$ formally represents a space-dependent Gaussian white noise on the open, bounded, smooth domain $G \subset \mathbb{R}^n$ for $n \leq 3$, and $\delta(x - x_0)$ is the Dirac distribution at $x_0 \in G$. The cost functional is

$$J(\omega_0, \omega_1, u, T) = \int_0^T (|\omega(t)|^2_{H^2(G)} + |\omega_t(t)|^2_{L^2(G)} + |u(t)|^2)dt \tag{4.17}$$

The deterministic version of this control problem, that is $\eta \equiv 0$, is given in [1], [12].

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