STABILIZATION OF GTF SYSTEMS WITH UNKNOWN PARAMETERS*

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Abstract. We solve the problem of global uniform input-to-state stabilization of general triangular form systems whose dynamics depends on unknown parameters and is affected by bounded disturbances. We suppose that both the unknown parameters and the external disturbances enter nonlinearly into the dynamics and that the dynamics is periodic in time. Also we assume that the unknown parameters are bounded by some known constants and that the equilibrium is a regular point of the system. The controller obtained depends on time and states only, i.e., we do not introduce any dynamic estimates of the unknown parameters in contrast to the classical designs of adaptive controllers.

Key words. ISS stabilization, robust and adaptive control, nonlinear control

AMS subject classifications. 93D21, 93D15

1. Introduction. Design of controllers, when the dynamics of a system has some unknown parameters (adaptive control) or it is affected by some external disturbances (robust control), is important in many applications. For the triangular form (TF) systems [1]

$$\dot{x}_1 = f_1(t, x_1, x_2),\ \ \dot{x}_2 = f_2(t, x_1, x_2, x_3), \dots, \dot{x}_n = f_n(t, x_1, \dots, x_n, u)$$

which appear in many engineering models, the backstepping designs [2, 3] became an efficient solution of the above-mentioned problem. However, their classical version was applicable to the so-called "regular case", when $|\frac{\partial f_i}{\partial x_{i+1}}| \neq 0$, $i = 1, \dots, n$ or more often when the system is in the strict-feedback form, i.e., $f_i(x_1, ..., x_{i+1}) = b_i x_{i+1} + \varphi_i(x_1, ..., x_i)$, with $b_i \neq 0$, $i = 1, \dots, n$. At the same time [4], or some later [5, 6], integrator backstepping was extended to the singular case, in which the conditions $|\frac{\partial f_i}{\partial x_{i+1}}| \neq 0$ does not necessarily hold. Most works dealing with the singular case are devoted either to the polynomial structure of $f_i(x_1, ..., x_{i+1})$ w.r.t. x_{i+1} [4, 6] or to the assumptions that $f_i(x_1, ..., x_{i+1})$ has upper and lower estimates by a polynomial form w.r.t. x_{i+1} [7] or to the case of local stabilization of triangular systems which satisfy some homogeneity conditions [4,5]. A more general situation is the so-called "generalized triangular form" (GTF), when the only assumption is that $f_i(x_1, ..., x_i, \cdot)$ are surjective, and x_i are vectors not necessarily of the same dimension. Until now, only controllability or stabilization problems were considered for the GTF control systems [8,9] while the controllers design problems when a system has unknown parameters has not been investigated yet. The goal of the current paper is to tackle this problem and to bridge this gap. The problem formulation in the current paper is motivated by [3] in which the dynamics depends on unknown parameter and is affected by external disturbances. However, in comparison with work [3]: (a) we deal with the GTF systems, while [3] is devoted to the strict-feedback forms, (b) in contrast to [3] we allow dynamics to be nonlinear w.r.t. the disturbances and w.r.t. the unknown parameters (note that the problem of adaptive control with nonlinear parametrization is meaningful even for the TF systems in the regular case, see [10]).

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2. Preliminaries. Recall that a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$, if it is continuous, strictly increasing and $\alpha(0) = 0$, and is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if for each fixed $t \geq 0$ the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed $s \geq 0$, we have $\beta(s, t) \to 0$ as $t \to +\infty$ and $t \mapsto \beta(s, t)$ is decreasing.

Consider the nonlinear system

$$\dot{x}(t) = F(t, x(t), D(t)), \qquad t \in \mathbb{R} \tag{2.1}$$

with states $x \in \mathbb{R}^n$, and external disturbances $D(\cdot)$ in $L_\infty(\mathbb{R}; \mathbb{R}^{q_0})$. We assume that F is continuous w.r.t (t, x, D) and Lipschitz continuous w.r.t. (x, D).

For every $D(\cdot) \in L_\infty(\mathbb{R}; \mathbb{R}^{q_0})$, we denote by $\|D(\cdot)\|$ its L_∞ - norm on $\mathbb{R}$, and for each $x^0 \in \mathbb{R}^n$, each $t_0 \in \mathbb{R}$ by $x(t, t_0, x^0, D(\cdot))$ we denote the solution of the Cauchy problem $x(t_0) = x^0$ for system (2.1) with this $D(\cdot)$. The following definition can be found in many works (at least for the equilibrium at $x^* = 0$) and is a natural extension the original notion of the ISS introduced by E.D.Sontag in [11].

Definition 2.1. *System (2.1) is said to be uniformly input-to-state stable (ISS) at point $x^* \in \mathbb{R}^n$ if there are $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for each $t_0 \in \mathbb{R}$ each $\xi \in \mathbb{R}^n$ and each $D(\cdot) \in L_\infty(\mathbb{R}; \mathbb{R}^{q_0})$ we have*

$$|x(t, t_0, \xi, D(\cdot)) - x^*| \leq \max\{\beta(|\xi - x^*|, t - t_0),\ \gamma(\| D(\cdot)\|_{L_\infty[t_0, +\infty[})\}, \quad t \geq t_0. \tag{2.2}$$

Note that, if (2.1) is uniformly ISS at $x^* \in \mathbb{R}^n$, then by this definition x^* is an equilibrium of (2.1) with $D(\cdot) = 0$. For any $N \in \mathbb{N}$ by $\langle \cdot, \cdot \rangle$ we denote the scalar product in $\mathbb{R}^N$ and for $\xi \in \mathbb{R}^N$ let $|\xi|$ denote its quadratic norm, i.e., $|\xi| = \langle \xi, \xi \rangle^{\frac{1}{2}}$.

3. Main result. Consider a system of the form

$$\left\{ \begin{array}{l} \dot{x}_1 = f_1(t, x_1, x_2) + \varphi_1(t, x_1, \theta, D(t)), \\ \dot{x}_2 = f_2(t, x_1, x_2, x_3) + \varphi_2(t, x_1, x_2, \theta, D(t)), \\ \dots \\ \dot{x}_\nu = f_\nu(t, x_1, \dots, x_\nu, u) + \varphi_\nu(t, x_1, \dots, x_\nu, \theta, D(t)), \end{array} \right. \qquad t \in \mathbb{R} \tag{3.1}$$

with state $x = [x_1^T, \dots, x_\nu^T]^T \in \mathbb{R}^n$, where $x_i \in \mathbb{R}^{m_i}$, $m_i \leq m_{i+1}$, $n = m_1 + \dots + m_\nu$, controls $u \in \mathbb{R}^{\mathrm{m}} = \mathbf{R}^{m_\nu}$, with external disturbances $D(\cdot) \in L_\infty(\mathbb{R}; \mathbb{R}^{q_0})$ and with unknown parameter θ. We suppose that (3.1) satisfies the following assumptions:

- A1: $f = (f_1, \dots, f_\nu)^T$ and $\varphi = (\varphi_1, \dots, \varphi_\nu)^T$ are of class $C^{\nu+1}$ and T-periodic in time, i.e., for some $T > 0$: $f(t + T, x, u) = f(t, x, u)$ and $\varphi(t + T, x, \theta, D) = \varphi(t, x, \theta, D)$ for all $[t, x, u, \theta, D]$.
- A2: For every $[x_1, \dots, x_i]$ in $\mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_i}$, every $t \in [0, T]$, and every $i = 1, \dots, \nu$, the map $x_{i+1} \mapsto f_i(t, x_1, \dots, x_i, x_{i+1}) : \mathbb{R}^{m_{i+1}} \to \mathbb{R}^{m_i}$ is a surjection, i.e., $f_i(t, x_1, \dots, x_i, \mathbb{R}^{m_{i+1}}) = \mathbb{R}^{m_i}$
- A3: $\exists x_i^* \in \mathbb{R}^{m_i}$, $1 \leq i \leq \nu$, and $u^* = x_{\nu+1}^* \in \mathbb{R}^{\mathrm{m}}$ s.t. $\forall t \in [0, T]$ $\forall i = 1, \dots, \nu$ we have $\operatorname{rank} \frac{\partial f_i}{\partial x_{i+1}}(t, x_1^*, \dots, x_{i+1}^*) = m_i$, and $f_i(t, x_1^*, ..., x_{i+1}^*) = \varphi_i(t, x_1^*, ..., x_i^*, \theta, 0) = 0 \in \mathbb{R}^{m_i}$ for all $t \in [0, T]$, $\theta \in \mathbb{R}^N$, $i = 1, \dots, \nu$

Theorem 3.1. *(Main result). Suppose that system (3.1) satisfies conditions A1-A3 and the unknown parameter θ satisfies $|\theta| \leq \theta^*$ with some* known *$\theta^* > 0$. Then (3.1) is globally uniformly input-to-state stabilizable by means of a C^1 time-varying T-periodic feedback law, i.e., there exists a feedback law $u(t, x)$ of class $C^1(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^{\mathrm{m}})$ s.t. $\forall [t, x] \in \mathbb{R} \times \mathbb{R}^n$ $u(t + T, x) = u(t, x)$ and $u(t, x^*) = u^*$, and s.t. system (3.1) with $u = u(t, x)$ is globally uniformly ISS at the equilibrium $x^* = (x_1^*, \dots, x_\nu^*)^T$.*

Let us note that our stabilizing feedback depends on (t, x) only, i.e., we do not introduce any dynamical variables like estimates of unknown θ etc. As we will see in the proof, it is possible, because being motivated by [3] we assumed above that *unknown* vector θ lies in some *known* compact subset of $\mathbb{R}^N$.

4. Example. Consider the system

$$\begin{cases} \dot{x}_1 = (y_3 - y_1 - 1)\cos y_2 - x_2^2 - 1 + x_2 D(\theta^2 + 1) \\ \dot{x}_2 = (y_3 - y_1 - 1)\sin y_2 - x_1^2 - 1 + x_1 D(\theta^2 + 2) \\ \dot{y}_1 = \mathrm{e}^{y_1} u_2 \\ \dot{y}_2 = u_3 \cos u_1 - \frac{1}{x_1^2 + 1} \\ \dot{y}_3 = u_3 \sin u_1 - \frac{1}{x_2^2 + 1}, \end{cases} \tag{4.1}$$

where $[x_1, x_2, y_1, y_2, y_3]^T \in \mathbb{R}^5$ is the state, $u = [u_1, u_2, u_3]^T \in \mathbb{R}^3$ is the control, $D \in \mathbb{R}^1$ is the external disturbance and $\theta \in \mathbb{R}^1$ is the unknown parameter such that $|\theta| \le \theta^*$ with some known $\theta^* > 0$. Every point $[x_1^*, x_2^*, y_1^*, y_2^*, y_3^*]$ such that $(y_3^* - y_1^* - 1)\cos y_2^* = {x_2^*}^2 + 1$ and $(y_3^* - y_1^* - 1)\sin y_2^* = {x_1^*}^2 + 1$ can be treated as an equilibrium with $D \equiv 0$ which satisfies condition A3 (note that such a point is always regular). Then system (4.1) satisfies A1-A3 and for any $T > 0$ there is a T -periodic feedback $u(t, x, y)$ of class C^1 such that system (4.1) with $u = u(t, x, y)$ is globally uniformly ISS with respect to the disturbance D. Let us remark that Theorem 3.1 does not address the problem of stabilization at a *singular* equilibrium point. For example, if one wants to stabilize system $\dot{x}_1 = x_2^3 + x_1$, $\dot{x}_2 = u$ at $[0, 0]$, then one needs another approach, because $[0, 0]$ is a singular point of the system (condition A3 does not hold true). As regards such polynomial forms with singular equilibrium point, we refer to [6] for example. On the other hand system (4.1) does not belong to the classes explored in [1–7, 10] and the methods proposed in these works are not applicable here.

5. Recursive design. We prove Theorem 1 by using a certain backstepping technique developed earlier in [9] for global asymptotic stabilization of ODE GTF systems. In the current paper, we show that this technique can be revised for the case of *adaptive* ISS stabilization as well. For this purpose, consider a control system of the form

$$\begin{cases} \dot{z} = g(t, z, w) + \Phi(t, z, \theta, D(t)) \\ \dot{w} = h(t, z, w, v) + \Psi(t, z, w, \theta, D(t)), \end{cases} \qquad t \in \mathbb{R} \tag{5.1}$$

with states $y = [z, w]^T \in \mathbb{R}^k \times \mathbb{R}^{\mathrm{q}_0}$, controls $v \in \mathbb{R}^m$, disturbances $D(\cdot) \in L_\infty(\mathbb{R}; \mathbb{R}^{\mathrm{l}_0})$ and unknown parameter $\theta \in \mathbb{R}^N$. We suppose that (5.1) satisfies the conditions:

(A1') $g \in C^{\mu+1}(\mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^{\mathrm{q}_0}; \mathbb{R}^k)$; $\Phi \in C^{\mu+1}(\mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^N \times \mathbb{R}^{\mathrm{l}_0}; \mathbb{R}^k)$; $h \in C^{\mu+1}(\mathbb{R} \times \mathbb{R}^{k+\mathrm{q}_0} \times \mathbb{R}^m; \mathbb{R}^{\mathrm{q}_0})$; $\Psi \in C^{\mu+1}(\mathbb{R} \times \mathbb{R}^{k+\mathrm{q}_0} \times \mathbb{R}^N \times \mathbb{R}^{\mathrm{l}_0}; \mathbb{R}^{\mathrm{q}_0})$; *and there is* $T > 0$ *s.t.* $\forall\, t \in \mathbb{R}\ \forall\, (z, w, v, \theta, D) \in \mathbb{R}^k \times \mathbb{R}^{\mathrm{q}_0} \times \mathbb{R}^m \times \mathbb{R}^N \times \mathbb{R}^{\mathrm{l}_0}$ *we have*

$$g(t+T, z, w) = g(t, z, w); \quad \Phi(t+T, z, \theta, D) = \Phi(t, z, \theta, D);$$

$$h(t+T, z, w, v) = h(t, z, w, v); \quad \Psi(t+T, z, w, \theta, D) = \Phi(t, z, w, \theta, D)$$

(A2') *For every* $t \in \mathbb{R}$ *and every* $\theta \in \mathbb{R}^N$ *we have* $\operatorname{rank} \frac{\partial h(t,0,0,0)}{\partial v} = \mathrm{q}_0 \le m$ *and*

$$g(t, 0, 0) = \Phi(t, 0, \theta, 0) = 0 \in \mathbb{R}^k, \quad h(t, 0, 0, 0) = \Psi(t, 0, 0, \theta, 0) = 0 \in \mathbb{R}^{\mathrm{q}_0}$$

(A3') *For every* $[t, z, w]$ *in* $[0, T] \times \mathbb{R}^{k+\mathrm{q}_0}$ *the map* $v \mapsto h(t, z, w, v)$ *is surjective, i.e.,* $h(t, z, w, \mathbb{R}^m) = \mathbb{R}^{\mathrm{q}_0}$.

Along with system (5.1) we consider its "reduced order subsystem"

$$\dot{z} = g(t, z, w) + \Phi(t, z, \theta, D(t)), \qquad t \in \mathbb{R}, \tag{5.2}$$

where $z \in \mathbb{R}^k$ is treated as the state, $w \in \mathbb{R}^{\mathrm{q}_0}$ is treated as the control; $\theta \in \mathbb{R}^N$, $D \in \mathbb{R}^{\mathrm{l}_0}$ are vectors of unknown parameters and external disturbances respectively.

Given any $D(\cdot) \in L_\infty(\mathbb{R}; \mathbb{R}^{\mathrm{l}_0})$, any $t_0 \in \mathbb{R}$, any $\theta \in \mathbb{R}^N$, any $y_0 \in \mathbb{R}^{k+\mathrm{q}_0}$, any $z_0 \in \mathbb{R}^k$, and any controllers $w(\cdot, \cdot)$ and $v(\cdot, \cdot, \cdot)$ for systems (5.2) and (5.1) respectively, let $t \mapsto z(t, t_0, z_0, w(\cdot, \cdot), \theta, D(\cdot))$ and $t \mapsto y(t, t_0, y_0, y(\cdot, \cdot, \cdot), \theta, D(\cdot))$ denote the trajectories, of systems (5.2) and (5.1) with these θ and $D(\cdot)$, defined respectively by the corresponding controls $w(\cdot, \cdot)$ and $v(\cdot, \cdot, \cdot)$ and by the initial conditions $z(t_0) = z_0$ and $y(t_0) = y^0$.

For systems (5.2) and (5.1) consider the following Lyapunov functions respectively

$$W(z) = \langle z, z \rangle \quad \text{ and } \quad V(y) = \langle y, y \rangle \quad \text{ for all } \; y = (z, w)^T \in \mathbb{R}^k \times \mathbb{R}^{\mathrm{q}_0}.$$

For the base case of the backstepping design, we consider a system of the form

$$\dot{w} = h(t, w, v) + \Psi(t, w, \theta, D(t)), \;\; t \in \mathbb{R} \tag{5.3}$$

with states $w = \in \mathbb{R}^{\mathrm{q}_0}$ and controls $v \in \mathbb{R}^m$ as a special case of systems of form (5.1) with *empty reduced order system (5.2)*, where functions $h(t, w, v)$, $\Psi(t, w, \theta, D)$ satisfy (A1')-(A3'). *For this case, we say by definition that the corresponding reduced order subsystem (5.2) is empty and satisfies Assumptions (A1')-(A3') by definition.*

THEOREM 5.1. *Assume that system (5.2) satisfies (A1')-(A3') and that for some sequences* $\{r_q\}_{q=2}^\infty \subset \mathbb{R}$, $\{\rho_q\}_{q=1}^\infty \subset \mathbb{R}$ *and* $\{d_q\}_{q=1}^\infty \subset \mathbb{R}$ *such that* $0 < \rho_q < r_{q+1} < \rho_{q+1} \to \infty$ *and* $0 < d_q < d_{q+1} \to \infty$ *as* $\mathbb{N} \ni q \to \infty$ *it holds that* $d_1 < \max_{|z| \le \rho_1} \lambda W(z)$ *with some* $\lambda > 0$ *and there are* $\gamma(\cdot) \in \mathcal{K}_\infty$ *and some known* $\theta^* > 0$ *such that*

(i) $\forall\, t \in [0, T]\; \forall\, z \in \mathbb{R}^k,\; \forall\, D \in \mathbb{R}^{\mathrm{l}_0}\; \forall\, \theta \in \mathbb{R}^N\; (\,|z| \le r_2,\; |\theta| \le \theta^*,\; \gamma(|D|) \le d_1) \;\Rightarrow$

$$\frac{\partial W(z)}{\partial z}(g(t, z, w)|_{w=0} + \Phi(t, z, \theta, D)) \le -\lambda W(z) + \gamma(|D|),$$

(ii) For every $q \in \mathbb{N}$ *and all* $z_0 \in \mathbb{R}^k$, $t_0 \in [0, T]$, $\theta \in \mathbb{R}^N$ *and* $D(\cdot) \in L_\infty([t_0, t_0 + T]; \mathbb{R}^{\mathrm{l}_0})$, *if* $|z_0| \le r_{q+2}$ *and* $|\theta| \le \theta^*$ *and* $\gamma(\| D(\cdot)\|_{L_\infty[t_0, t_0+T]}) \le d_q$, *then*

$$|z(t, t_0, z_0, 0, \theta, D(\cdot))|^2 \le \rho_{q+2}^2 - \frac{t - t_0}{T}(\rho_{q+2}^2 - \rho_q^2) \;\; t \in [t_0, t_0 + T].$$

Then, for every $\varepsilon \in]0, \lambda[$ *there exist an integer* $q_0 \ge 0$, $q_0 \in \mathbb{Z}$, *positive real* r_1, r_0, $\ldots$, r_{-q_0}, d_0, d_{-1} $\ldots$, d_{-q_0-1} *and a sequence* $\{R_q\}_{q=-q_0-1}^{+\infty} \subset]0, +\infty[$ *such that* $d_{-q_0-1} < (\lambda - \varepsilon) \max_{|y| \le R_{-q_0-1}} V(y)$ *and* $0 < R_q < r_{q+1} < R_{q+1} \to +\infty$ *and* $0 < d_q < d_{q+1}$ *for all* $q \ge -q_0 - 1$, $q \in \mathbb{Z}$, *and there is a feedback* $v(\cdot, \cdot)$ *of class* $C^\mu(\mathbb{R} \times \mathbb{R}^{k+\mathrm{q}_0}; \mathbb{R}^m)$ *and a gain* $\hat{\gamma}(\cdot) \ge \gamma(\cdot)$ *of class* $\mathcal{K}_\infty$ *such that* $v(t, 0) = 0 \in \mathbb{R}^m$ *and* $v(t + T, y) = v(t, y)$ *for all* $[t, y] \in \mathbb{R} \times \mathbb{R}^{k+\mathrm{q}_0}$ *and*

(I) For every $y = (z, w)^T \in \mathbb{R}^{k+\mathrm{q}_0}$, *every* $D \in \mathbb{R}^{\mathrm{l}_0}$ *and every* $\theta \in \mathbb{R}^N$, *if* $|y| \le r_{-q_0}$ *and* $\hat{\gamma}(|D|) \le d_{-q_0-1}$ *and* $|\theta| \le \theta^*$, *then*

$$\frac{\partial V(z, w)}{\partial z}\,(g(t, z, w) + \Phi(t, z, \theta, D)) + \frac{\partial V(z, w)}{\partial z}\,(h(t, z, w, v(t, z, w))$$

$$+\Psi(t, z, w, \theta, D)) \le -(\lambda - \varepsilon) V(y) + \hat{\gamma}(|D|)$$

(II) For all $q \geq -q_0 - 1$, $y_0 \in \mathbb{R}^{k+\mathrm{q}_0}$, $t_0 \in \mathbb{R}$, $\theta \in \mathbb{R}^N$ *and* $D(\cdot) \in L_\infty([t_0, t_0 + T]; \mathbb{R}^{\mathrm{l}_0})$, *if* $|y_0| \leq r_{q+2}$ *and* $\hat{\gamma}(\| D(\cdot)\|_{L_\infty[t_0,t_0+T]}) \leq d_q$, *and* $|\theta| < \theta^*$ *then*

$$|y(t, t_0, y_0, v(\cdot,\cdot), \theta, D(\cdot))|^2 \leq R_{q+2}^2 - \frac{t - t_0}{T}(R_{q+2}^2 - R_q^2), \quad \text{for all } t \in [t_0, t_0 + T]$$

If system (5.1) has form (5.3) with reduced order subsystem (5.2) and satisfies (A1')-(A3'), then all the assumptions of this Theorem are supposed to be true by definition with any sequences $0 < \rho_q < r_{q+1} < \rho_{q+1} \to \infty$ *and* $0 < d_q < d_{q+1} \to \infty$, *any* $\lambda > 0$ *and any* $\gamma(\overline{D}) = \gamma^* \overline{D}^2$ $(\gamma^* > 0)$. *In this case, by definition, the Theorem states that there is a feedback* $v(\cdot,\cdot)$ *for system (5.3) such that all the stated above including (I),(II) holds for system (5.3) with this feedback* $v(\cdot,\cdot)$.

6. Proofs of Theorems 3.1 and 5.1. First we show that Theorem 5.1 implies Theorem 3.1. Then we show how to prove Theorem 5.1.

Proof. [of Theorem 3.1] Without loss of generality assume that $x^* = 0$, $u^* = 0$. The backstepping design is by induction on $p = 0, 1, \ldots, \nu - 1$, where p is the number of first equations in (3.1).

For $p = 0$, we define $w := x_1$, $v := x_2$, $h(t, w, v) := f_1(t, x_1, x_2)$, $\Psi(t, w, \theta, D) := \varphi(t, w, \theta, D)$; $\mathrm{q}_0 := m_1$, $\mathrm{m} := m_2$ and consider system (5.1), which is the same as (5.3). Applying Theorem 5.1, we find a T - periodic, $C^{\nu+1}$ - smooth feedback $v = x_2 = \alpha_1(t, x_1) = \alpha_1(t, w)$ which stabilizes (5.3) in the sense of Theorem 5.1 and satisfies (I)-(II) with any $\lambda > 0$. Then we define: $z := x_1$, $w = x_2 - \alpha_1(t, x_1)$, $v := x_3$ and

$$g(t, z, w) := f_1(t, z, w + \alpha_1(t, z)), \quad \Phi(t, z, \theta, D) := \varphi(t, z, \theta, D),$$

$$h(t, z, w, v) := f_2(t, z, w + \alpha_1(t, z), v) - \frac{\partial \alpha_1(t, z)}{\partial t} - \frac{\partial \alpha_1(t, z)}{\partial z} f_1(t, z, w + \alpha_1(t, z))$$

$$\Psi(t, z, w, \theta, D) := -\frac{\partial \alpha_1(t, z)}{\partial z} \varphi_1(t, z, \theta, D) + \varphi_2(t, z, w + \alpha_1(t, z), \theta, D) \tag{6.1}$$

For system (5.1) defined by (6.1), we apply Theorem 5.1 again. Subsequently, arguing by induction over $p = 0, 1, \ldots, \nu - 1$, we obtain global diffeomorphism of states which brings the dynamics of (3.1) to that of (5.1) and such that (5.1) satisfies statement of Theorem 5.1 including conditions (I),(II). Let us prove that the obtaiined feedback $v(t, Y)$ provides uniform ISS property of system (5.1), then the inverse transformation provides the same property for (3.1), which completes the proof of Theorem 3.1.

Using property (I), we extend sequences $\{d_q\}_{q=-q_0-1}^{+\infty}$, $\{r_q\}_{q=-q_0}^{+\infty}$, $\{R_q\}_{q=-q_0-1}^{+\infty}$ to the whole $\mathbb{Z}$ by choosing some positive $\{d_q\}_{q=-q_0-2}^{-\infty}$, $\{r_q\}_{q=-q_0-1}^{-\infty}$, $\{R_q\}_{q=-q_0-2}^{-\infty}$ such that $d_{q+1} > d_q \to +0$ and $r_{q+1} > R_q > r_q \to +0$ as $q \to -\infty$ and such that the feedback $v(\cdot,\cdot)$ obtained and the sequences $\{d_q\}_{q=-\infty}^{+\infty} \subset]0, +\infty[$, $\{r_q\}_{q=-\infty}^{+\infty} \subset]0, +\infty[$ and $\{R_q\}_{q=-\infty}^{+\infty} \subset]0, +\infty[$ satisfy the conditions:

(a) $0 < r_q < R_q < r_{q+1} < R_{q+1}$ and $0 < d_q < d_{q+1}$ for all $q \in \mathbb{Z}$.

(b) $r_q \to 0$, $R_q \to 0$, $d_q \to 0$, as $q \to -\infty$ and $r_q \to +\infty$, $R_q \to +\infty$, $d_q \to +\infty$, as $q \to +\infty$.

(c) For each $y_0 \in \mathbb{R}^{k+\mathrm{q}}$, $t_0 \in \mathbb{R}$, $\theta \in \mathbb{R}^N$ and each $D(\cdot) \in L_\infty([t_0, t_0 + T]; \mathbb{R}^{l_0})$, if $|y_0|^2 \leq r_{q+2}^2$ and $\gamma_{ext}(\| D(\cdot)\|_{L_\infty[t_0,t_0+T]}) \leq d_q$ and $|\theta| \leq \theta^*$ then

$$\forall t \in [t_0, t_0 + T] \qquad |y(t, t_0, y_0, v(\cdot,\cdot), \theta, D(\cdot))|^2 \leq R_{q+2}^2 - \frac{t - t_0}{T}(R_{q+2}^2 - R_q^2)$$

Let functions $\overline{\beta}(\cdot,\cdot) \in \mathcal{KL}$ and $\overline{\gamma}(\cdot) \in \mathcal{K}_\infty$ be defined as follows: $\beta(0,\tau) = \gamma(0) := 0$ for all $\tau \ge 0$ and

$$\overline{\beta}(r_{q+1},\tau) := R^2_{(q-\kappa)+2} - \frac{\tau - \kappa T}{T}(R^2_{(q-\kappa)+2} - R^2_{(q-\kappa)+1}),$$

$$\text{whenever } \tau \in [\kappa T, (\kappa+1)T[, \quad \kappa \in \mathbb{Z}_+ \text{ and,} \tag{6.2}$$

$$\overline{\beta}(r,\tau) := \beta(r_q,\tau) + \frac{r - r_q}{r_{q+1} - r_q}(\overline{\beta}(r_{q+1},\tau) - \overline{\beta}(r_q,\tau)),$$

$$\text{whenever } r \in [r_q, r_{q+1}[, \quad \text{with } q \in \mathbb{Z} \quad \text{for all } \tau \ge 0, \text{ and} \tag{6.3}$$

$$\overline{\gamma}(\overline{D}) := R^2_{q+2} + \frac{\overline{D} - \hat{\gamma}^{-1}(d_{q-1})}{\hat{\gamma}^{-1}(d_q) - \hat{\gamma}^{-1}(d_{q-1})}(R^2_{q+3} - R^2_{q+2}),$$

$$\text{whenever } \overline{D} \in [\hat{\gamma}^{-1}(d_{q-1}), \hat{\gamma}^{-1}(d_q)[, \tag{6.4}$$

Then, using (a), (b) (c), it is not difficult to prove that for every $t_0 \in \mathbb{R}$, every $\kappa \in \mathbb{Z}_+$, every $y_0 \in \mathbb{R}^{k+\mathrm{q}}$, every $D(\cdot)$ in $L_\infty(\mathbb{R};\mathbb{R}^{l_0})$ and every $\theta \in \mathbb{R}^N$ such that $|\theta| \le \theta^*$ we obtain

$$|y(t,t_0,y_0,v(\cdot,\cdot),\theta,D(\cdot))|^2 \le \max\{\overline{\beta}(|y_0|, t-t_0), \overline{\gamma}(\| D(\cdot)\|_{L_\infty[t_0,t_0+\kappa T]})\}$$

$$\le \max\{\overline{\beta}(|y_0|, t-t_0), \overline{\gamma}(\| D(\cdot)\|_{L_\infty(\mathbb{R};\mathbb{R}^{\mathrm{l}_0})})\} \qquad \text{for all } \ t \in [t_0, t_0+\kappa T].$$

Since $\kappa \in \mathbb{Z}_+$ is arbitrary, the latter inequality holds for all $t \ge t_0$. This completes the proof of Theorem 3.1. □

Proof. [of Theorem 5.1] The proof is only sketched due to space limits. We revise the approach applied in the proof of Theorem 3.1 in [9] to the case of asymptotic stabilization. Let us show how to revise Step 1 in Section 5 of [9]. By the assumptions of Theorem 5.1, if $|z| \le r_2$, $|\theta| \le \theta^*$, $\gamma(|D|) \le d_1$, then

$$\dot{V}|_{(5.1),v} = 2\langle z, g(t,z,0) + \Phi(t,z,\theta,D)\rangle + 2\langle z, g(t,z,w) - g(t,z,0)\rangle + 2\langle w, h(t,z,w,v)$$

$$+\Psi(t,z,w,\theta,D)\rangle \le -\lambda\langle z,z\rangle + \gamma(|D|) + 2\langle w, h(t,z,w,v) + J^*(t,z,w)z + \Psi(t,z,w,\theta,D)\rangle,$$

where $J(t,z,w) = \int_0^1 \frac{\partial g}{\partial w}(t,z,sw)ds$. By the well-known Hadamard lemma, we obtain $\langle w, \Psi(t,y,\theta,D)\rangle = \sum\limits_{i=1}^{\mathrm{q}_0} w_i \Psi_i(t,y,\theta,D)$, where

$$\Psi_i(t,y,\theta,D) = \sum_{j=1}^{k} \psi_{i,j}(t,y,\theta,D)z_j + \sum_{j=1}^{\mathrm{q}_0} \phi_{i,\mathrm{q}}(t,y,\theta,D)w_\mathrm{q} + \sum_{\mathrm{l}=1}^{\mathrm{l}_0} \chi_{i,\mathrm{l}}(t,y,\theta,D)D_\mathrm{l},$$

for all $(t,y,\theta,D) = (t,z,w,\theta,D) \in \mathbb{R} \times \mathbb{R}^{k+\mathrm{q}_0} \times \mathbb{R}^{N+\mathrm{l}_0}$ with

$$\psi_{i,j}(t,y,\theta,D) = \int\limits_0^1 \frac{\partial \Psi_i}{\partial z_j}(t,sy,\theta,sD)ds, \quad \phi_{i,\mathrm{q}}(t,y,\theta,D) = \int\limits_0^1 \frac{\partial \Psi_i}{\partial w_\mathrm{q}}(t,sy,\theta,sD)ds$$

$$\chi_{i,\mathrm{l}}(t,y,\theta,D)=\int_0^1\frac{\partial\Psi_i}{\partial D_\mathrm{l}}(t,sy,\theta,sD)ds,$$

for all $j=1,\dots,k$, $\mathrm{q}=1,\dots,\mathrm{q}_0$, $\mathrm{l}=1,\dots,\mathrm{l}_0$; $i=1,\dots,\mathrm{q}_0$. Then, for any $\varepsilon^*>0$ and any $\gamma^*>0$, if $|y|\le r_2,\ |\theta|\le\theta^*,\ \gamma(|D|)\le d_1$, then

$$2\langle w,\Psi(t,y,\theta,D)\rangle=2\sum_{i=1}^{\mathrm{q}_0}w_i\left(\sum_{j=1}^{k}\psi_{i,j}(t,y,\theta,D)z_j+\sum_{\mathrm{q}=1}^{\mathrm{q}_0}\phi_{i,\mathrm{q}}(t,y,\theta,D)w_\mathrm{q}\right.$$

$$\left.+\sum_{\mathrm{l}=1}^{\mathrm{l}_0}\chi_{i,\mathrm{l}}(t,y,\theta,D)D_\mathrm{l}\right)\le M\left(\sum_{i=1}^{\mathrm{q}_0}\sum_{j=1}^{k}\left(\frac{|w_i|^2}{\varepsilon^*}+\varepsilon^*|z_j|^2\right)\right.$$

$$\left.+\sum_{i=1}^{\mathrm{q}_0}\sum_{\mathrm{q}=1}^{\mathrm{q}_0}\left(|w_i|^2+|w_\mathrm{q}|^2\right)+\sum_{i=1}^{\mathrm{q}_0}\sum_{l=1}^{\mathrm{l}_0}\left(\frac{|w_i|^2}{\gamma^*}+\gamma^*|D_\mathrm{l}|^2\right)\right)$$

with

$$M_{i,j,\mathrm{q},\mathrm{l}}:=\max_{\overline{\mathcal{B}}}(|\psi_{i,j}(t,z,w,\theta,D)|+|\phi_{i,\mathrm{q}}(t,z,w,\theta,D)|+|\chi_{i,\mathrm{l}}(t,z,w,\theta,D)|)$$

and

$$M:=\max\{\,M_{i,j,\mathrm{q},\mathrm{l}}\mid i=1,\dots,\mathrm{q}_0,\ j=1,\dots,k,\ \mathrm{q}=1,\dots,\mathrm{q}_0,\ \mathrm{l}=1,\dots,\mathrm{l}_0\},$$

where

$$\overline{\mathcal{B}}:=\{(t,z,w,\theta,D)\mid t\in[0,T],\ |z|^2+|w|^2\le r_2^2,\ |\theta|\le\theta^*,\ |D|\le\gamma^{-1}(d_1)\}.$$

Thus in $\overline{\mathcal{B}}$

$$\dot V|_{(5.1)}\le-(\lambda-M\varepsilon^*\mathrm{q}_0)\langle z,z\rangle+2\langle w,h(t,y,v)+J^*(t,y)z+\frac{1}{2}F(y)\rangle$$

$$+\gamma(|D|)+\gamma^*M\mathrm{q}_0|D|^2,$$

where

$$F(y)=\left(\frac{kM}{\varepsilon^*}+2M\mathrm{q}_0+\frac{M\mathrm{l}_0}{\gamma^*}\right)w.$$

Finally we use Assumption (A2') and the implicit function theorem and obtain the existence of $r>0$ and a T - periodic feedback $v(t,y)$ of class C^μ such that

$$2h(t,y,v(t,y))=-(\lambda-M\varepsilon^*\mathrm{q}_0)w-2J^*(t,y)z-F(y),\qquad t\in\mathbb{R},$$

whenever $|y|^2=|z|^2+|w|^2\le 2r$. This yields property (I) stated in Theorem 5.1 and completes the revision of Step 1 in [9] Section 5. Then in Steps 2-5 one should amend successively formulas (20),(24),(39), the definition of $\Lambda_{q+1}(M_1)$ at p.1800, formulas (41),(45),(50),(59)-(64),(73) and the proof of Lemma 5.1 with (74)-(75) at p. 1804 by adding $\Phi(t,z,\theta,D)$ and $\Psi(t,y,\theta,D)$ into all the estimates accordingly. □

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