ON A DISCRETE-TIME H^∞ CONTROL PROBLEM WITH ADDITIONAL INPUT NON-MINIMUM PHASE PROPERTY

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Abstract. An H^∞ control problem for a class of discrete-time systems with additional non-minimum phase property at the control input is addressed. First, the full information problem is dealt with by developing a discrete-time counterpart of the J-spectral factorization technique in the literature. Second, the output estimation problem is reduced to finite-dimensional one by examining the J-spectral factorization technique. The resulting H^∞ controller is constructed from the standard KYP equations and implemented in the Smith form.

Key words. J-spectral factorization, Non-minimum phase property, Smith predictor

1. Introduction. A class of infinite-dimensional H^∞ control problems is studied under the setup in Fig. 1.1, where P_+ is a controlled plant which consists of a rational P and a general scalar inner function m which represents additional non-minimum phase property, and K_+ is a controller to design. As the problem is very simple and also includes important classes of time delay systems, many constructive solution methods have been studied. Among them, the technique of reducing an irrational J-spectral density to rational one is initially presented to solve a continuous-time input-delayed mixed sensitivity problem in [3]. In [4], [5], this technique is enhanced to treat the continuous-time H^∞ problem with m in Fig. 1.1 being a delay function. In [5], the resulting H^∞ controller is parameterized with the Riccati equation in the standard H^∞ problem (when $m(s) = 1$). The structure of them is identified as the Smith-type one which consists of the finite-dimensional H^∞ controller for the unstable dynamics and the measurement compensation part based on the past history of the control input. However, it is not a trivial question whether the discrete-time H^∞ controller can be also parameterized with the standard Riccati or KYP equation and implemented in the Smith form. This is because the discrete-time KYP equation has more complicated structures than the continuous-time counterpart. The approach in [4], [5] involves the preliminary step where the inner function at the control input is absorbed temporarily into the controller to design. The step requires auxiliary arguments to guarantee the causality of the controller, and elaborate manipulation of transfer functions to obtain the final form of it. In [2], it is shown that the J-spectral factorization technique is also applicable to the continuous-time H^∞ problem with m in Fig. 1.1 being a general continuous-time inner function. The approach in [2] does not involves the above preliminary step, but requires the regularity of the direct feedthrough matrices of P.

This paper pursues a parameterization of the discrete-time H^∞ suboptimal controllers. A discrete-time counterpart of the J-spectral factorization technique in [4] is developed as the main tool. In contrast to [2], [4], [5], the first step is to simplify the original problem to the full information one, and to apply the J-spectral factorization technique to the latter. The second step is to formulate an output estimation problem in order to realize the H^∞ state-feedback law based on partial information. Focusing on the relationship between the state variables of the irrational and reduced rational

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J-spectral densities, the measured output is modified in order to reduce the output estimation problem to finite-dimensional one. Such a modification of the measured output is also successful in the corresponding H^2 problem [1]. The approach taken in this paper gives a parameterization of the discrete-time H^∞-suboptimal controllers using the standard KYP equation. They are implemented in the Smith form as shown in Fig. 1.2.

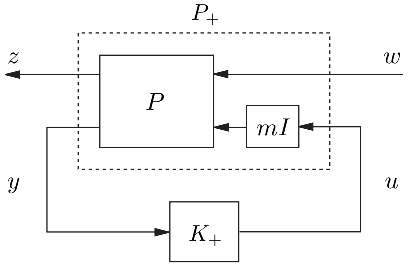

FIG. 1.1. *Control system*

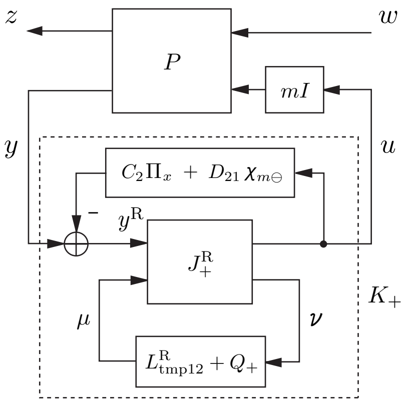

FIG. 1.2. *H^∞ controller in Smith form*

This paper is organized as follows: In Section 2, the problem formulation and assumptions are stated. In Section 3, the full information problem where the state and disturbance are available is tackled with the proposed discrete-time J-spectral factorization technique. In Section 4, it is noted how to solve the output feedback problem from the result of the full information problem. In Section 5, the proofs left in Section 3 are given.

Notations: For a system $P : (w,\, u) \to (z,\, y)$, the scattering representation $\mathcal{C}\,(P)$ is defined as the mapping from $(y,\, u)$ to $(w,\, z)$. The inverse mapping of $\mathcal{C}\,(\cdot)$ is denoted by $\mathcal{C}^{-1}\,(\cdot)$. For a system $\Phi : (y,\, u) \to (h,\, k)$, the Schur complementation transform $\mathcal{S}(\Phi)$ [4] is defined as the mapping from $(h,\, u)$ to $(-y,\, k)$. The inverse mapping of $\mathcal{S}(\cdot)$ is denoted by $\mathcal{S}^{-1}(\cdot)$. The $(i,\, j)$ block of a matrix M is denoted by M_{ij}. The maximal eigenvalue of a square matrix M is denoted by $\lambda_{\max}\,(M)$. The transfer function from the input b to the output a is denoted by $T_{a\,b}$. Finally, $J_{\mathrm{s}} := \begin{bmatrix} O & -I \\ I & O \end{bmatrix}$ is a block matrix which defines a symplectic inner product.

2. Problem formulation. Our objectives are to derive a solvability condition of the discrete-time H^∞ problem for P_+, of which the rational part P is given by $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & O & D_{12} \\ C_2 & D_{21} & O \end{array}\right]$, and to clarify the structure of the H^∞ controller K_+, which renders the L^2 gain from w to z less than γ. The parameter λ denotes any positive number greater than or equal to the upper bound γ. In what follows, subscripts are used to show whether matrices and operators depend on γ or on λ. If no confusion occurs, the subscript of γ may be dropped. The solvability condition is given under the following conditions (X1)-(X3) and (Y). For simplicity, the following definition is used: $\begin{bmatrix} Q & S_2 \\ S_2^* & R_2 \end{bmatrix} := \begin{bmatrix} C_1 & D_{12} \end{bmatrix}^* \begin{bmatrix} C_1 & D_{12} \end{bmatrix}$.

(X1) In the case $m(z) = 1$, the H^∞ full information problem is solvable. Therefore, for $\forall\lambda \geq \gamma$, the KYP equation

$$F_\lambda^* R_{\mathrm{mm\,c}\,\lambda} F_\lambda = Q + A^* X_\lambda A - X_\lambda, \quad -R_{\mathrm{mm\,c}\,\lambda} F_\lambda = S^* + B^* X_\lambda A,$$
$$R_{\mathrm{mm\,c}\,\lambda} = R_{\mathrm{mm}\,\lambda} + B^* X_\lambda B,$$
$$B := \begin{bmatrix} B_1 & B_2 \end{bmatrix}, \; S := \begin{bmatrix} O & S_2 \end{bmatrix}, \; R_{\mathrm{mm}\,\lambda} := \begin{bmatrix} -\lambda^2 I & O \\ O & R_2 \end{bmatrix}$$

has the positive semidefinite stabilizing solution X_λ such that $A_{\mathrm{c}\,\lambda} := A + B_1 F_{1\,\lambda} + B_2 F_{2\,\lambda}$ is stable, and $R_{\mathrm{mm\,c}\,\lambda}$ satisfies the following definiteness conditions:

$$R_{\mathrm{mm\,c}\,22\,\lambda} > O, \; -\lambda^2 \Lambda_{\mathrm{c}\,\lambda} := R_{\mathrm{mm\,c}\,11\,\lambda} - R_{\mathrm{mm\,c}\,12\,\lambda} R_{\mathrm{mm\,c}\,22\,\lambda}^{-1} R_{\mathrm{mm\,c}\,21\,\lambda} < O.$$

(X2) A is invertible, and hence the symplectic matrix $H_{\mathrm{nc}\,\lambda}$ can be defined by $H_{\mathrm{nc}\,\lambda} := H_{\mathrm{nc}\,\delta\,\lambda}^{-1} H_{\mathrm{nc}\,\sigma} = \begin{bmatrix} A - \frac{1}{\lambda^2} B_1 B_1^* A^{-*} Q & -\frac{1}{\lambda^2} B_1 B_1^* A^{-*} \\ A^{-*} Q & A^{-*} \end{bmatrix}$, where

$$H_{\mathrm{nc}\,\delta\,\lambda} = \begin{bmatrix} I & \frac{1}{\lambda^2} B_1 B_1^* \\ O & A^* \end{bmatrix}, \; H_{\mathrm{nc}\,\sigma} := \begin{bmatrix} A & O \\ Q & I \end{bmatrix}.$$

(X3) For $\forall\lambda \geq \gamma$, the eigenvalues of $H_{\mathrm{nc}\,\lambda}$ and the singular points of $m(z)$ are disjoint to each other, and hence the symplectic matrix $\Sigma_\lambda := m^\sim(H_{\mathrm{nc}\,\lambda})$ is defined by the Cauchy integral formula.

(Y) In the case $m(z) = 1$, the H^∞ full control problem is solvable. Therefore, the KYP equation has the positive semidefinite stabilizing solution Y and gain $L =: \begin{bmatrix} L_1 & L_2 \end{bmatrix}$ such that $\acute{A}_{\mathrm{c}} := A + L_1 C_1 + L_2 C_2$ is stable.

3. Full information problem. We focus on the full information problem and assume that the internal state and exogenous disturbance are available. Let $y_{\mathrm{mm}} := w$ be the available disturbance, then the transfer function $P_{+\,\mathrm{mm}}$ from $(w,\, u)$ to $(z,\, y_{\mathrm{mm}})$ is given by $P_{+\,\mathrm{mm}} = P_{\mathrm{mm}} \begin{bmatrix} I & \\ & mI \end{bmatrix}$, $P_{\mathrm{mm}} := \begin{bmatrix} P_{11} & P_{12} \\ I & O \end{bmatrix}$. If the original H^∞ problem is solvable, then for $\forall\lambda \geq \gamma$, the following model matching problem should also be solvable:

(MM) Find the transfer function $T_{u\,w} \in H^\infty$ such that $P_{11} + P_{12} m\, T_{u\,w} \in H^\infty$ and $\|P_{11} + P_{12} m\, T_{u\,w}\|_\infty < \lambda$.

(MM) is recast to the following factorization problem (F) or J-spectral factorization problem (JSF):

(F) Find appropriate intermediate signals $(\widetilde{w},\, \widetilde{u})$ and the factorization of $\mathcal{C}\,(P_{+\,\mathrm{mm}})$ such that $\mathcal{C}\,(P_{+\,\mathrm{mm}}) = N_{+\,\lambda} M_{+\,\lambda}^{-1}$ holds, where $N_{+\,\lambda} : (\widetilde{w},\, \widetilde{u}) \to (w,\, z)$ is a J-inner function and $M_{+\,\lambda} : (\widetilde{w},\, \widetilde{u}) \to (y_{\mathrm{mm}},\, u)$ is a stable function with strictly causal $\left(M_{+\,\lambda}^{-1}\right)_{12}$.

(JSF) Define the J-spectral density $\Phi_{+\,\lambda} : (y_{\mathrm{mm}},\, u) \to (h_{\mathrm{mm}},\, k)$ by the equation $\Phi_{+\,\lambda} := \mathcal{C}\,(P_{+\,\mathrm{mm}})^\sim \begin{bmatrix} -\lambda^2 I & \\ & I \end{bmatrix} \mathcal{C}\,(P_{+\,\mathrm{mm}})$. Find the J-spectral factorization of it, which satisfies (J1) and (J2):

(J1) There exist positive definite matrices $\Lambda_{\mathrm{c}\,\lambda}^{\mathrm{R}}$, $R_{\mathrm{mm\,c}\,22\,\lambda}^{\mathrm{R}}$ and a stable J-spectral factor $M_{+\,\lambda}$ with strictly causal $\left(M_{+\,\lambda}^{-1}\right)_{12}$ such that

$$\Phi_{+\,\lambda} = M_{+\,\lambda}^{-\sim} \begin{bmatrix} -\lambda^2 \Lambda_{\mathrm{c}\,\lambda}^{\mathrm{R}} & \\ & R_{\mathrm{mm\,c}\,22\,\lambda}^{\mathrm{R}} \end{bmatrix} M_{+\,\lambda}^{-1}. \tag{3.1}$$

(J2) $N_{+\,\lambda} := \mathcal{C}\,(P_{+\,\mathrm{mm}})\, M_{+\,\lambda}$ is a J-inner function.

The main purpose of this section is to solve the problem (JSF) for $\lambda = \gamma$, or equivalently for $\forall\lambda \geq \gamma$. By analyzing the structure of $\Phi_{+\,\lambda}$, the J-spectral factorization of it is reduced to that of some rational spectral density $\Phi^{\mathrm{R}}_{+\,\lambda}$. This reduction technique can be seen as a discrete-time counterpart of [4].

Before proceeding, the following *discrete-time* completion operator is introduced. It has properties similar to the *continuous-time* counterpart in [4], [2].

DEFINITION 3.1. *For a given discrete-time inner function $m(z)$ and state-space system $G(z) = \left[\begin{array}{c|c} A & B \\ \hline C & CA^{-1}B \end{array}\right]$, the discrete-time completion operator is defined by*

$$\chi_m\left[G\right](z) := G_{\chi m}(z) - m(z)G(z),\ G_{\chi m}(z) := \left[\begin{array}{c|c} A & m(A)B \\ \hline C & Cm(A)A^{-1}B \end{array}\right].$$

Define the following causal and stable function via the discrete-time truncation operator: $\chi_{m\,\lambda}(z) := \chi_m\left[\left(P_{11}^{\sim}P_{11} - \lambda^2 I\right)^{-1} P_{11}^{\sim}P_{12}\right](z)$, and decompose it into the constant matrix $\chi_{m0\,\lambda}$ and the strictly causal function $\chi_{m\ominus\lambda}(z)$: $\chi_{m\,\lambda}(z) = \chi_{m0\,\lambda} + \chi_{m\ominus\lambda}(z)$. It is noted that $\chi_{m0\,\lambda}$ is given by $\chi_{m0\,\lambda} = \frac{1}{\lambda^2}B_1^{*}A^{-*}\left(S^{\mathrm{R}}_{2\,\lambda} - S_2 m(\infty)\right)$, where $\begin{bmatrix} B^{\mathrm{R}}_{2\lambda} \\ S^{\mathrm{R}}_{2\,\lambda}\end{bmatrix} := \Gamma_{\lambda}^{-1}\begin{bmatrix} B_2 \\ S_2\end{bmatrix}$ and $\Gamma_{\lambda} := H_{\mathrm{nc}\,\delta\,\lambda}\Sigma_{\lambda}H^{-1}_{\mathrm{nc}\,\delta\,\lambda} = H_{\mathrm{nc}\,\sigma}\Sigma_{\lambda}H^{-1}_{\mathrm{nc}\,\sigma}$. As shown in Lemma 3.2, the irrational $\Phi_{+\,\lambda}$ is linked with the rational $\Phi^{\mathrm{R}}_{+\,\lambda}$ using the strictly causal part of $\chi_{m\,\lambda}(z)$.

LEMMA 3.2. *The following J-spectral density*

$$\Phi^{\mathrm{R}}_{+\,\lambda} := \begin{bmatrix} I & \chi_{m\ominus\lambda} \\ O & I \end{bmatrix}^{\sim} \Phi_{+\,\lambda}\begin{bmatrix} I & \chi_{m\ominus\lambda} \\ O & I \end{bmatrix}. \tag{3.2}$$

is rational. The state variables $(x,\,p)$ of Φ_{+} and $(x^{\mathrm{R}},\,p^{\mathrm{R}})$ of Φ^{R}_{+} are related as follows:

$$\begin{bmatrix} x(z)^T & p(z)^T \end{bmatrix}^T = \begin{bmatrix} x^{\mathrm{R}}(z)^T & p^{\mathrm{R}}(z)^T \end{bmatrix}^T + \Pi_{\lambda}(z)u(z), \tag{3.3}$$

where $\Pi_{\lambda}(z) := \begin{bmatrix} \Pi_{x\,\lambda}(z) \\ \Pi_{p\,\lambda}(z)\end{bmatrix} := \left(zI - H_{\mathrm{nc}\,\lambda}\right)^{-1}\left(m(z)I - \Sigma_{\lambda}^{-1}\right)H^{-1}_{\mathrm{nc}\,\delta\,\lambda}\begin{bmatrix} B_2 \\ S_2\end{bmatrix}$.

Proof. Let $\Omega_{+} := \mathcal{S}(\Phi_{+})$, and define the rational Ω^{R}_{+} by the following state-space realization

$$\begin{bmatrix} x^{\mathrm{R}}(n+1) \\ p^{\mathrm{R}}(n+1)\end{bmatrix} = H_{\mathrm{nc}\,\lambda}\begin{bmatrix} x^{\mathrm{R}}(n) \\ p^{\mathrm{R}}(n)\end{bmatrix} + H^{-1}_{\mathrm{nc}\,\delta\,\lambda}\begin{bmatrix} -\frac{1}{\lambda^2}B_1 & B^{\mathrm{R}}_{2\lambda} \\ O & S^{\mathrm{R}}_{2\,\lambda}\end{bmatrix}\begin{bmatrix} h_{\mathrm{mm}}(n) \\ u(n)\end{bmatrix}$$

$$\begin{aligned}\begin{bmatrix} -y^{\mathrm{R}}_{\mathrm{mm}}(n) \\ k^{\mathrm{R}}(n)\end{bmatrix} &= \begin{bmatrix} -\frac{1}{\lambda^2}B_1 & B^{\mathrm{R}}_{2\lambda} \\ O & S^{\mathrm{R}}_{2\,\lambda}\end{bmatrix}^{*} H^{-*}_{\mathrm{nc}\,\sigma}J_{\mathrm{s}}\begin{bmatrix} x^{\mathrm{R}}(n) \\ p^{\mathrm{R}}(n)\end{bmatrix} \\ &\quad + \begin{bmatrix} -\frac{1}{\lambda^2}I & \frac{1}{\lambda^2}B_1^{*}A^{-*}S^{\mathrm{R}}_{2\,\lambda} - \chi_{m0\,\lambda} \\ -\chi^{*}_{m0\,\lambda} & R_2 - B_2^{*}A^{-*}S_2\end{bmatrix}\begin{bmatrix} h_{\mathrm{mm}}(n) \\ u(n)\end{bmatrix}.\end{aligned}$$

By comparing the realizations of Ω_{+} and Ω^{R}_{+}, the following properties are observed; 1) $\begin{bmatrix} x(z)^T & p(z)^T\end{bmatrix}^T - \begin{bmatrix} x^{\mathrm{R}\,T}(z) & p^{\mathrm{R}\,T}(z)\end{bmatrix}^T$ is independent of $h_{\mathrm{mm}}(z)$ and determined by $u(z)$, namely Eq. (3.3) holds; 2) $m^{\sim}(z)\begin{bmatrix} x(z)^T & p(z)^T\end{bmatrix}^T - \Sigma_{\lambda}\begin{bmatrix} x^{\mathrm{R}}(z)^T & p^{\mathrm{R}}(z)^T\end{bmatrix}^T$ is independent of $u(z)$ and determined by $h_{\mathrm{mm}}(z)$. These properties yield the identity: $\Omega^{\mathrm{R}}_{+\,\lambda} = \Omega_{+\,\lambda} + \begin{bmatrix} O & \chi_{m\ominus\lambda} \\ \chi^{\sim}_{m\ominus\lambda} & O\end{bmatrix}$. Taking the inverse Schur transformation $\mathcal{S}^{-1}(\cdot)$ of this identity, we have Eq. (3.2). ☐

By Lemma 3.2, it is necessary for (J1) that there exists a stable J-spectral factor $M^{\mathrm{R}}_{+\,\lambda}$ with strictly causal $\left(M^{\mathrm{R}\,(-1)}_{+\,\lambda}\right)_{12}$ such that

$$\Phi^{\mathrm{R}}_{+\,\lambda} = M^{\mathrm{R}\,(-\sim)}_{+\,\lambda}\begin{bmatrix} -\lambda^2\Lambda^{\mathrm{R}}_{\mathrm{c}\,\lambda} & \\ & R^{\mathrm{R}}_{\mathrm{mm\,c}\,22\,\lambda}\end{bmatrix} M^{\mathrm{R}\,(-1)}_{+\,\lambda}. \tag{3.4}$$

This is equivalent to the following conditions (C1) and (C2).

(C1) The KYP equation

$$F_\lambda^{\mathrm{R}*} R_{\mathrm{mm\,c}\,\lambda}^{\mathrm{R}} F_\lambda^{\mathrm{R}} = Q + A^* X_\lambda^{\mathrm{R}} A - X_\lambda^{\mathrm{R}},$$

$$-R_{\mathrm{mm\,c}\,\lambda}^{\mathrm{R}} F_\lambda^{\mathrm{R}} = \begin{bmatrix} O & S_{2\lambda}^{\mathrm{R}} \end{bmatrix}^* + \begin{bmatrix} B_1 & B_{2\lambda}^{\mathrm{R}} - B_1 \chi_{m0\,\lambda} \end{bmatrix}^* X_\lambda^{\mathrm{R}} A,$$

$$R_{\mathrm{mm\,c}\,\lambda}^{\mathrm{R}} = R_{\mathrm{mm}\,\lambda}^{\mathrm{R}} + \begin{bmatrix} B_1 & B_{2\lambda}^{\mathrm{R}} - B_1 \chi_{m0\,\lambda} \end{bmatrix}^* X_\lambda^{\mathrm{R}} \begin{bmatrix} B_1 & B_{2\lambda}^{\mathrm{R}} - B_1 \chi_{m0\,\lambda} \end{bmatrix}, \tag{3.5}$$

where $R_{\mathrm{mm}\,\lambda}^{\mathrm{R}} := \begin{bmatrix} -\lambda^2 I & \lambda^2 \chi_{m0\,\lambda} \\ \lambda^2 \chi_{m0\,\lambda}^* & R_{\mathrm{mm}\,22\,\lambda}^{\mathrm{R}} \end{bmatrix}$ and $R_{\mathrm{mm}\,22\,\lambda}^{\mathrm{R}} := R_2 - B_2^* A^{-*} S_2 + B_{2\lambda}^{\mathrm{R}*} A^{-*} S_{2\lambda}^{\mathrm{R}} - \lambda^2 \chi_{m0\,\lambda}^* \chi_{m0\,\lambda}$, has the stabilizing solution X_λ^{R} such that $A_{\mathrm{c}\,\lambda}^{\mathrm{R}} := A + B_1 F_{1\lambda}^{\mathrm{R}} + \left(B_{2\lambda}^{\mathrm{R}} - B_1 \chi_{m0\,\lambda}\right) F_{2\lambda}^{\mathrm{R}}$ is stable.

(C2) $R_{\mathrm{mm\,c}\,\lambda}^{\mathrm{R}}$ in Eq. (3.5) satisfies the following definiteness conditions:

$$R_{\mathrm{mm\,c}\,22\,\lambda}^{\mathrm{R}} > O,\ -\lambda^2 \Lambda_{\mathrm{c}\,\lambda}^{\mathrm{R}} := R_{\mathrm{mm\,c}\,11\,\lambda}^{\mathrm{R}} - R_{\mathrm{mm\,c}\,12\,\lambda}^{\mathrm{R}} R_{\mathrm{mm\,c}\,22\,\lambda}^{\mathrm{R}\,(-1)} R_{\mathrm{mm\,c}\,21\,\lambda}^{\mathrm{R}} < O.$$

The following lemma represents the solution of the KYP equation in (C1) using that of the KYP equation in (X1).

LEMMA 3.3. *For $\exists \lambda \geq \gamma$, suppose that the following conditions are satisfied; 1) the KYP equation in (X1) has the stabilizing solution X_λ; 2) $\Sigma_\lambda = m^{\sim}(H_{\mathrm{nc}\,\lambda})$ is defined. Then, the condition (C1) is equivalent to the following condition (C3):*

(C3) The (2, 2) block of the symplectic matrix $\Sigma_{X\,\lambda} := \begin{bmatrix} I & O \\ X_\lambda & I \end{bmatrix} \Sigma_\lambda \begin{bmatrix} I & O \\ -X_\lambda & I \end{bmatrix}$, namely, $\Sigma_{X\,22\,\lambda} = \Sigma_{22\,\lambda} + X_\lambda \Sigma_{12\,\lambda}$ is regular.

If the above condition is satisfied, X_λ^{R}, $A_{\mathrm{c}\,\lambda}^{\mathrm{R}}$ and $F_\lambda^{\mathrm{R}} =: \begin{bmatrix} F_{1\lambda}^{\mathrm{R}\,T} & F_{2\lambda}^{\mathrm{R}\,T} \end{bmatrix}^T$ are given by

$$X_\lambda^{\mathrm{R}} = X_\lambda + \Sigma_{X\,22\,\lambda}^{-1} \Sigma_{X\,21\,\lambda} = \left(X_\lambda \Sigma_{12\,\lambda} + \Sigma_{22\,\lambda}\right)^{-1} \left(X_\lambda \Sigma_{11\,\lambda} + \Sigma_{21\,\lambda}\right), \tag{3.6}$$

$$A_{\mathrm{c}\,\lambda}^{\mathrm{R}} = \Sigma_{X\,22\,\lambda}^* A_{\mathrm{c}\,\lambda} \Sigma_{X\,22\,\lambda}^{-*},\ F_{1\lambda}^{\mathrm{R}} = \frac{1}{\lambda^2} B_1^* X_\lambda^{\mathrm{R}} A_{\mathrm{c}\,\lambda}^{\mathrm{R}} + \chi_{m0\,\lambda} F_{2\lambda}^{\mathrm{R}},\ F_{2\lambda}^{\mathrm{R}} = F_{2\lambda} \Sigma_{X\,22\,\lambda}^{-*}. \tag{3.7}$$

Proof. See Subsection 5.1. ☐

The following theorem gives a constructive solution to (JSF), and the solutions to (MM) and (F) can be obtained from it.

THEOREM 3.4. *Suppose that (X1)-(X3) are satisfied. The J-spectral factorization of $\Phi_{+\gamma}$ satisfying (J1)-(J2) exists if and only if (C2)-(C3) are satisfied for $\forall \lambda \geq \gamma$. Under the existence conditions, $M_{+\lambda}$ in (J1) and $N_{+\lambda}$ in (J2) are given as follows:*

$$M_{+\lambda} = \begin{bmatrix} I & \chi_{m\ominus\,\lambda} \\ O & I \end{bmatrix} M_{+\lambda}^{\mathrm{R}},\ \text{where} \tag{3.8}$$

$$M_{+\lambda}^{\mathrm{R}} := \left[\begin{array}{c|cc} A_{\mathrm{c}\,\lambda}^{\mathrm{R}} & B_1 & B_{2\lambda}^{\mathrm{R}} - B_1 \chi_{m0\,\lambda} \\ \hline F_{1\lambda}^{\mathrm{R}} & I & O \\ F_{2\lambda}^{\mathrm{R}} & O & I \end{array}\right] \begin{bmatrix} I & O \\ F_{21\,\lambda}^{\mathrm{R}} & I \end{bmatrix},\ F_{21\,\lambda}^{\mathrm{R}} := -R_{\mathrm{mm\,c}\,22\,\lambda}^{\mathrm{R}\,(-1)} R_{\mathrm{mm\,c}\,21\,\lambda},$$

and $N_{+\lambda} = N_\lambda N_{+\lambda}^m$, where

$$N_\lambda := \left[\begin{array}{c|cc} A_{\mathrm{c}\,\lambda} & B_1 & B_2 \\ \hline F_{1\,\lambda} & I & O \\ C_1 + D_{12} F_{2\,\lambda} & O & D_{12} \end{array}\right] \begin{bmatrix} I & O \\ F_{21\,\lambda} & I \end{bmatrix},\ F_{21\,\lambda} := -R_{\mathrm{mm\,c}\,22\,\lambda}^{-1} R_{\mathrm{mm\,c}\,21\,\lambda},$$

$$N_{+\lambda}^m := \begin{bmatrix} I & O \\ -F_{21\,\lambda} & I \end{bmatrix} \left\{ - \begin{bmatrix} \frac{1}{\lambda^2} B_1^* A^{-*} Q + F_{1\,\lambda} & \frac{1}{\lambda^2} B_1^* A^{-*} \\ F_{2\,\lambda} & O \end{bmatrix} \left(m(z) I - \Sigma_\lambda^{-1}\right) \right. \tag{3.9}$$

$$\left. \cdot \left(zI - H_{\mathrm{nc}\,\lambda}\right)^{-1} \begin{bmatrix} (*1) & (*2) \end{bmatrix} + \begin{bmatrix} I & -\frac{1}{\lambda^2} B_1^* A^{-*} S_2 \left(m(z) - m(\infty)\right) \\ O & m(z) I \end{bmatrix} \right\} \begin{bmatrix} I & O \\ F_{21\,\lambda}^{\mathrm{R}} & I \end{bmatrix},$$

$$(*1) = -\begin{bmatrix} I \\ -X_\lambda \end{bmatrix} \Sigma_{X\,22\,\lambda}^{-*} B_1,\ (*2) = H_{\mathrm{nc}\,\delta\,\lambda}^{-1} \begin{bmatrix} B_2 \\ S_2 \end{bmatrix} - \begin{bmatrix} I \\ -X_\lambda \end{bmatrix} \Sigma_{X\,22\,\lambda}^{-*} \left(B_{2\lambda}^{\mathrm{R}} - B_1 \chi_{m0\,\lambda}\right).$$

Proof. By Lemma 3.2, the conditions (C1) and (C2) are necessary, and $M_{+\lambda}$ is constructed as above. By Lemma 3.3, (C1) is replaced with (C3). For sufficiency, $N_{+\lambda} := \mathcal{C}\left(P_{\mathrm{mm}}\right) M_{+\lambda}$ is proved to be a J-inner function in Subsection 5.2. ☐

4. Output estimation problem. We focus on that $x(z)$ is decomposed into the sum of $x^{\mathrm{R}}(z)$ and $\Pi_{x\,\lambda}(z)u(z)$ in Eq. (3.3). Instead of x, x^{R} is estimated based on the original measured output y below. Referring to Theorem 3.4, let

$$y^{\mathrm{R}}_{\mathrm{mm}}(n) := y_{\mathrm{mm}}(n) - \chi_{m\ominus}(z)u(n) = F^{\mathrm{R}}_1 x^{\mathrm{R}}(n) + \widetilde{w}(n), \tag{4.1}$$

$P_{+\,\mathrm{stb}} := \mathcal{C}^{-1}(N_+)$, $P_{+\,\mathrm{mm\,tmp}} := \mathcal{C}^{-1}\left(M_+^{-1}\right)$, and $P^{\mathrm{R}}_{+\,\mathrm{mm\,tmp}} := \mathcal{C}^{-1}\left(M_+^{\mathrm{R}\,(-1)}\right)$. Then $P_{+\,\mathrm{mm}}$ is factorized as shown in Fig. 4.1. Moreover, $P_{+\,\mathrm{stb}}$ is a J-lossless system since N_+ is a J-inner function. Let the plant $P_{+\,\mathrm{tmp}} : (\widetilde{w},\, u) \to (\widetilde{u},\, y)$ of which the measured output is defined as y instead of y_{mm} of $P_{+\,\mathrm{mm\,tmp}} : (\widetilde{w},\, u) \to (\widetilde{u},\, y_{\mathrm{mm}})$. From Eq. (3.3) and (4.1), y is represented by the state-variable x^{R} and measured output $y^{\mathrm{R}}_{\mathrm{mm}}$ of $P^{\mathrm{R}}_{+\,\mathrm{mm\,tmp}}$ as follows (see also Fig. 4.2):

$$\begin{aligned} y(n) &= C_2 x^{\mathrm{R}}(n) + D_{21} y^{\mathrm{R}}_{\mathrm{mm}}(n) + \{C_2\Pi_x(z) + D_{21}\chi_{m\ominus}(z)\}\, u(n) \\ &= \left(C_2 + D_{21}F^{\mathrm{R}}_1\right) x^{\mathrm{R}}(n) + D_{21}\widetilde{w}(n) + \{C_2\Pi_x(z) + D_{21}\chi_{m\ominus}(z)\}\, u(n). \end{aligned} \tag{4.2}$$

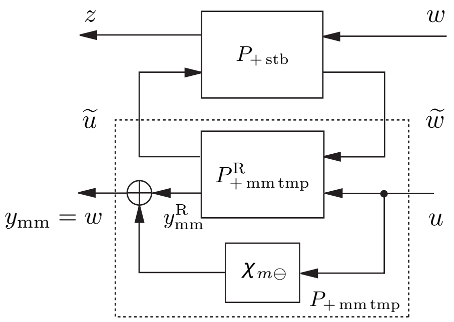

FIG. 4.1. *Factorization of* $P_{+\,\mathrm{mm}}$

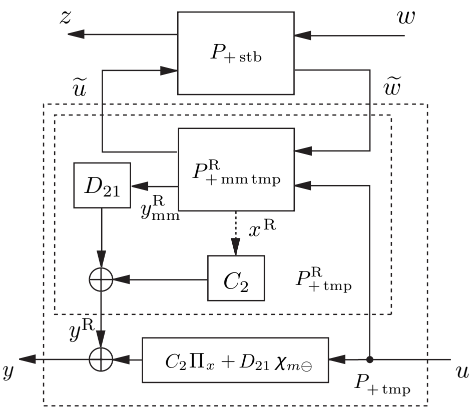

FIG. 4.2. *Representation of* y

Therefore, P_+ is factorized as $P_+ = P_{+\,\mathrm{stb}} \star P_{+\,\mathrm{tmp}}$. By Redheffer's lemma, the H^∞ control problem for P_+ is reduced to that for $P_{+\,\mathrm{tmp}}$. From Eq. (4.2), it is seen that $P_{+\,\mathrm{tmp}}$ consists of the finite-dimensional part:

$$P^{\mathrm{R}}_{+\,\mathrm{tmp}} = \left[\begin{array}{c|cc} A + B_1F^{\mathrm{R}}_1 & B_1 & B^{\mathrm{R}}_2 - B_1\chi_{m0} \\ \hline -F^{\mathrm{R}}_2 & -F^{\mathrm{R}}_{21} & I \\ C_2 + D_{21}F^{\mathrm{R}}_1 & D_{21} & O \end{array}\right] : (\widetilde{w},\, u) \to (\widetilde{u},\, y^{\mathrm{R}})$$

in output estimation form and the infinite-dimensional part:

$$y(n) = y^{\mathrm{R}}(n) + \{C_2\Pi_x(z) + D_{21}\chi_{m\ominus}(z)\}\, u(n). \tag{4.3}$$

Since the transfer function multiplied to $u(n)$ in Eq. (4.3) is strictly causal and stable, the output estimation problem for $P_{+\,\mathrm{tmp}}$ is reduced to that for $P^{\mathrm{R}}_{+\,\mathrm{tmp}}$ by modifying the measured output from y to y^{R}. The latter problem is finite-dimensional one, and its solvability can be stated using the following conditions.

(Z) $Z^{\mathrm{R}} := I - \frac{1}{\gamma^2}YX^{\mathrm{R}}$ satisfies the eigenvalue condition $\lambda_{\max}\left(Z^{\mathrm{R}}\right) < \gamma^2$.

(R) $\acute{R}^{\mathrm{R}}_{\mathrm{tmp\,c}} := \acute{D}^{\mathrm{R}}_{\tau}\begin{bmatrix} -\gamma^2\acute{\Lambda}^{\mathrm{R}}_{\mathrm{tmp}} & \\ & \acute{W}^{\mathrm{R}}_{\mathrm{tmp}} \end{bmatrix}\acute{D}^{\mathrm{R}*}_{\tau}$, where $\acute{D}^{\mathrm{R}}_{\tau} := \begin{bmatrix} I & -F^{\mathrm{R}}_{21} \\ O & D_{21} \end{bmatrix}$, $\acute{\Lambda}^{\mathrm{R}}_{\mathrm{tmp}} := R^{\mathrm{R}\,(-1)}_{\mathrm{mm\,c}\,22}$, $\acute{W}^{\mathrm{R}}_{\mathrm{tmp}} := \Lambda^{\mathrm{R}\,(-1)}_{\mathrm{c}}$, satisfies the following definiteness conditions:

$$\acute{R}^{\mathrm{R}}_{\mathrm{tmp\,c}\,22} > O,\ -\gamma^2\acute{\Lambda}^{\mathrm{R}}_{\mathrm{tmp\,c}} := \acute{R}^{\mathrm{R}}_{\mathrm{tmp\,c}\,11} - \acute{R}^{\mathrm{R}}_{\mathrm{tmp\,c}\,12}\acute{R}^{\mathrm{R}\,(-1)}_{\mathrm{tmp\,c}\,22}\acute{R}^{\mathrm{R}}_{\mathrm{tmp\,c}\,21} < O.$$

Parameterizing the H^∞ controller K_+^{R} for the finite-dimensional $P_{+\,\mathrm{tmp}}^{\mathrm{R}}$, we obtain the following theorem.

THEOREM 4.1. *Under the assumptions (X1)-(X3) and (Y), the H^∞ controller K_+ for P_+ exists if and only if (Z), (R), and for $\forall\lambda \geq \gamma$, (C2)-(C3) are satisfied. If the existence conditions are satisfied, $K_+ : (y,\, \mu) \to (u,\, \nu)$ has the structure of the Smith predictor shown in Fig. 1.2, which consists of the finite-dimensional part $K_+^{\mathrm{R}} : (y^{\mathrm{R}},\, \mu) \to (u,\, \nu)$ and the measurement compensation part in Eq. (4.3). K_+^{R} is parameterized by $K_+^{\mathrm{R}} = \mathcal{F}_l(J_+^{\mathrm{R}},\, L_{\mathrm{tmp12}}^{\mathrm{R}} + Q_+(z))$ for $\forall Q_+(z) \in H^\infty$ such that $\left\| \acute{\Lambda}_{\mathrm{tmp\,c}}^{\mathrm{R}\,(-1/2)} Q_+ \acute{R}_{\mathrm{tmp\,c\,22}}^{\mathrm{R}\,(1/2)} \right\|_\infty < \gamma$, where $L_{\mathrm{tmp12}}^{\mathrm{R}} := -\acute{R}_{\mathrm{tmp\,c\,12}}^{\mathrm{R}} \acute{R}_{\mathrm{tmp\,c\,22}}^{\mathrm{R}\,(-1)}$ and*

$$J_+^{\mathrm{R}} := \left[\begin{array}{c|cc} A_{\mathrm{c}}^{\mathrm{R}} + L_{\mathrm{tmp2}}^{\mathrm{R}}\left(C_2 + D_{21}F_1^{\mathrm{R}}\right) & -L_{\mathrm{tmp2}}^{\mathrm{R}} & L_{\mathrm{tmp1}}^{\mathrm{R}} \\ \hline F_2^{\mathrm{R}} & O & I \\ -\left(C_2 + D_{21}F_1^{\mathrm{R}}\right) & I & O \end{array}\right],$$

$$L_{\mathrm{tmp1}}^{\mathrm{R}} := Z^{\mathrm{R}\,(-1)}\left(B_2^{\mathrm{R}} + \frac{1}{\gamma^2}\acute{A}_{\mathrm{c}} Y S_2^{\mathrm{R}} + L_2 D_{21}\chi_{m0}\right),\ L_{\mathrm{tmp2}}^{\mathrm{R}} := Z^{\mathrm{R}\,(-1)} L_2.$$

5. Proofs.

5.1. Proof of Lemma 3.3. The KYP equation in (C1) is rewritten as follows:

$$\Phi_{+\,\sigma\,\lambda}^{\mathrm{R}} \mathcal{B}_\lambda^{\mathrm{R}} = \Phi_{+\,\delta\,\lambda}^{\mathrm{R}} \mathcal{B}_\lambda^{\mathrm{R}} A_{\mathrm{c}\,\lambda}^{\mathrm{R}},\ \mathcal{B}_\lambda^{\mathrm{R}} := \left[\begin{array}{cc|cc} I^T & -X_\lambda^{\mathrm{R}\,T} & F_{1\,\lambda}^{\mathrm{R}\,T} & F_{2\,\lambda}^{\mathrm{R}\,T} \end{array}\right]^T, \tag{5.1}$$

where $\Phi_{+\,\delta\,\lambda}^{\mathrm{R}} := \left[\begin{array}{cc|c} I & O & O \\ O & A^* & O \\ \hline O & (\dagger 1)^* & O \end{array}\right]$, $\Phi_{+\,\sigma\,\lambda}^{\mathrm{R}} := \left[\begin{array}{cc|c} A & O & (\dagger 1) \\ Q & I & (\dagger 2) \\ \hline (\dagger 2)^* & O & R_{\mathrm{mm}\,\lambda}^{\mathrm{R}} \end{array}\right]$,
$(\dagger 1) = \begin{bmatrix} B_1 & B_{2\,\lambda}^{\mathrm{R}} - B_1\chi_{m0\,\lambda} \end{bmatrix}$, $(\dagger 2) = \begin{bmatrix} O & S_{2\,\lambda}^{\mathrm{R}} \end{bmatrix}$. After a series of equivalence transformations to Eq. (5.1), we have

(5.2)
$$\Phi_{\sigma\,\lambda}^\times \mathcal{B}_\lambda^\times = \Phi_{\delta\,\lambda}^\times \mathcal{B}_\lambda^\times A_{\mathrm{c}\,\lambda}^{\mathrm{R}},\ \mathcal{B}_\lambda^\times := \left[\begin{array}{cc|cc} \alpha_\lambda^T & -\beta_\lambda^T & (\phi_\lambda - F_{1\,\lambda}\alpha_\lambda)^T & \left(F_{2\,\lambda}^{\mathrm{R}} - F_{2\,\lambda}\alpha_\lambda\right)^T \end{array}\right]^T,$$

where $\Phi_{\delta\,\lambda}^\times := \left[\begin{array}{cc|c} I & O & O \\ O & A_{\mathrm{c}\,\lambda}^* & O \\ \hline O & B^* & O \end{array}\right]$, $\Phi_{\sigma\,\lambda}^\times := \left[\begin{array}{cc|c} A_{\mathrm{c}\,\lambda} & O & B \\ O & I & O \\ \hline O & O & R_{\mathrm{mm\,c}\,\lambda} \end{array}\right]$,
$\begin{bmatrix} \alpha_\lambda \\ -\beta_\lambda \end{bmatrix} := \Sigma_{X\,\lambda} \begin{bmatrix} I \\ X_\lambda - X_\lambda^{\mathrm{R}} \end{bmatrix}$, $\phi_\lambda := F_{1\,\lambda}^{\mathrm{R}} + \frac{1}{\lambda^2}B_1^* A^{-*} S_2 m(\infty) F_{2\,\lambda}^{\mathrm{R}} + \frac{1}{\lambda^2}B_1^* A^{-*}\left(Q - X_\lambda^{\mathrm{R}}\right) - \frac{1}{\lambda^2}B_1^* A^{-*} Q\alpha_\lambda + \frac{1}{\lambda^2}B_1^* A^{-*}\left(X_\lambda\alpha_\lambda + \beta_\lambda\right) - \frac{1}{\lambda^2}B_1^* A^{-*} S_2 F_{2\,\lambda}^{\mathrm{R}}$. From the second row of Eq. (5.2), we have $\beta_\lambda - A_{\mathrm{c}\,\lambda}^* \beta_\lambda A_{\mathrm{c}\,\lambda}^{\mathrm{R}} = O$. Since both $A_{\mathrm{c}\,\lambda}$ and $A_{\mathrm{c}\,\lambda}^{\mathrm{R}}$ are stable, the following equality is obtained.

$$-\beta_\lambda = \Sigma_{X\,21\,\lambda} + \Sigma_{X\,22\,\lambda}\left(X_\lambda - X_\lambda^{\mathrm{R}}\right) = O. \tag{5.3}$$

Suppose that $\Sigma_{X\,22\,\lambda}$ is not regular, then there exists $v \neq 0$ such that $v^*\Sigma_{X\,22\,\lambda} = 0$. Pre-multiplying v^* to Eq. (5.3), we have $v^*\Sigma_{X\,21\,\lambda} = 0$, therefore $\begin{bmatrix} 0 & v^* \end{bmatrix}\Sigma_{X\,\lambda} = 0$. This contradicts the regularity of $\Sigma_{X\,\lambda}$. Consequently, $\Sigma_{X\,22\,\lambda}$ is regular. By Eq. (5.3), X_λ^{R} is given by Eq. (3.6) and α_λ is equal to $\Sigma_{X\,\lambda\,22}^{-*}$. Then, Eq. (3.7) is obtained.

5.2. Proof of Theorem 3.4. From Eq. (3.2) and (3.4), $\Phi_{+\,\lambda}$ is factorized as in Eq. (3.1) with $M_{+\,\lambda}$ in Eq. (3.8). Therefore, $N_{+\,\lambda}$ satisfies the identity

$$N_{+\,\lambda}^\sim \begin{bmatrix} -\lambda^2 I & \\ & I \end{bmatrix} N_{+\,\lambda} = \begin{bmatrix} -\lambda^2 \Lambda_{\mathrm{c}\,\lambda}^{\mathrm{R}} & \\ & R_{\mathrm{mm\,c\,22}\,\lambda}^{\mathrm{R}} \end{bmatrix}. \tag{5.4}$$

On the other hand, using $M_\lambda := \left[\begin{array}{c|cc} A_{c\,\lambda} & B_1 & B_2 \\ \hline F_{1\,\lambda} & I & O \\ F_{2\,\lambda} & O & I \end{array}\right] \begin{bmatrix} I & O \\ F_{21\,\lambda} & I \end{bmatrix}$, $N_{+\,\lambda}$ is rewritten as $N_{+\,\lambda} = \mathcal{C}\,(P_{\rm mm})\, M_\lambda \cdot M_\lambda^{-1} \begin{bmatrix} I & \chi_{m\ominus\lambda} \\ O & mI \end{bmatrix} M^{\rm R}_{+\,\lambda} = N_\lambda \cdot N^m_{+\,\lambda}$. This fact is implied in [6]. Note that N_λ is the J-inner function which appears in the standard H^∞ problem (the case $m(z) = 1$). In the following lemma, $N^m_{+\,\lambda}$ is shown to be a J-inner function as well.

LEMMA 5.1. *For $\forall\,\lambda \ge \gamma$, $N^m_{+\,\lambda}$ is a J-inner function.*

Proof. We argue similarly to the proof of Theorem 5.3 in [3]. From Eq. (5.4), $N^m_{+\,\lambda}$ is the J-unitary function which satisfies

$$N^{m\,\sim}_{+\,\lambda} \begin{bmatrix} -\lambda^2 \Lambda_{c\,\lambda} & \\ & R_{{\rm mm\,c}\,22\,\lambda} \end{bmatrix} N^m_{+\,\lambda} = \begin{bmatrix} -\lambda^2 \Lambda^{\rm R}_{c\,\lambda} & \\ & R^{\rm R}_{{\rm mm\,c}\,22\,\lambda} \end{bmatrix}. \tag{5.5}$$

To show that it is a J-inner function, it suffices to show that $G_\lambda(z) := N^m_{+\,11\,\lambda}(z)$ is bi-stable for $\forall\lambda \ge \gamma$. From Eq. (3.9), G_λ is written as $G_\lambda = I + \widetilde{G}_\lambda$, where $\widetilde{G}_\lambda$ is a stable function such that $\left\|\widetilde{G}_\lambda\right\|_\infty \to 0$ as $\lambda \to \infty$. Therefore, by the small gain theorem there exists $\exists\, c > 0$ such that for $\forall\,\lambda > c$, G_λ^{-1} is stable. Next, suppose that for $\gamma \le \exists\,\lambda_0 \le c$, $G_{\lambda_0}^{-1}$ is unstable. Then, by Nyquist's theorem there exists $\exists\,\theta_0$ such that $\det G_{\lambda_0}(e^{j\theta_0}) = 0$. Hence, for $\exists\, v \ne 0$, $G_{\lambda_0}(e^{j\theta_0})v = 0$. On the other hand, from the (1, 1) block in Eq. (5.5), the following equality holds.

$$-\lambda_0^2 G^*_{\lambda_0}(e^{j\theta_0}) \Lambda_{c\,\lambda_0} G_{\lambda_0}(e^{j\theta_0}) + N^{m\,*}_{+\,21\,\lambda_0}(e^{j\theta_0}) R_{{\rm mm\,c}\,22\,\lambda_0} N^m_{+\,21\,\lambda_0}(e^{j\theta_0}) = -\lambda_0^2 \Lambda^{\rm R}_{c\,\lambda_0}.$$

Multiplying v^* and v from the left and right, respectively, we have a contradiction. □

Consequently, $N_{+\,\lambda}$ is a J-inner function since it is the products of the J-inner functions N_λ and $N^m_{+\,\lambda}$.

6. Conclusion. The discrete-time H^∞ control problem with a general scalar inner function at the control input is addressed. A discrete-time counterpart of the J-spectral factorization technique in [3] is developed as the main tool. Focusing on the internal variables of the J-spectral densities, we proposed a new closed-loop reduction approach, and revealed that the discrete-time H^∞ controller is constructed from the standard KYP equations and implemented in the Smith form as in the continuous-time setting [5].

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