CASCADED FLIESS OPERATORS WITH ROUGH PATH INPUTS*

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Abstract. The lack of a suitable probabilistic characterization of the input process for a system of cascaded analytic nonlinear input-output maps is an obstacle to well-posedness. The only known condition is when a certain independence property is preserved by the first system in the connection. In this paper, T. Lyons' construction of a rough path is employed as an alternative characterization of an input process having arbitrary p-variation to ensure well-posedness of a cascade connection.

1. Introduction. In many applications, input-output systems are interconnected to form more complex systems. Describing the nature of the composite system and providing some explicit parametrization for it are generally nontrivial problems when the subsystems are nonlinear. The systems of interest here belong to the class of analytic nonlinear integral operators known as *Fliess operators* [5, 8, 13]. It was recently shown in [2, 3] that the lack of a suitable probabilistic characterization of the input process for interconnections of such systems is an obstacle to well-posedness. For example, the cascade connection of two Fliess systems is only known to be well-posed when a certain independence property is preserved by the first system in the connection. Hence, it appears that some alternative characterization of an input process is needed in this setting. One possibility is to employ T. Lyons' construction of a *rough path* [6, 11, 12]. This concept employs p-variation paths and Chen's identity in order to extend the notion of integration with respect to paths having finite p-variation for $p \geq 1$. It will be shown in this paper that the primary advantage of such an approach is that independence is no longer needed for producing well-posed cascaded Fliess operators.

The specific goals of the paper are to introduce the *Fliess signature operator*, which maps the input *signature* of a p-variation input to its corresponding output signature, and then to characterize the cascade interconnection for this class of operators. The first step is to introduce the notion of a path's signature as a substitute for its functional form. Lyon's rough path theory is then applied directly to characterize such a signature as a *p-rough path*. The next step is to introduce an algebraic device known as a *transduction* in order to define the Fliess signature operator. Finally, a characterization of the cascade interconnection is made in terms of transductions. In this setting, it can be shown directly that the composite system produces a well-defined output path, thus solving the open problem.

The paper is organized as follows. Section 2 summarizes the basics of rough path theory and Fliess operator theory used throughout the paper. In Section 3, the definitions of a Fliess operator and a Fliess signature operator driven by a p-rough path are presented. It is then shown that the output of a Fliess signature operator is a p-rough path. Finally, Section 4 gives a characterization of the cascade of Fliess signature operators in terms of the composition of their corresponding transductions. Given the

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space limitations, many of the proofs will be deferred to a future publication.

2. Fliess Operators with Rough Path Inputs. A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \dots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. The *length* of η, $|\eta|$, is the number of letters in η. The set of all words with length k is denoted by X^k. The set of all words including the empty word, $\emptyset$, is written as X^*. Clearly X^* forms a monoid under catenation. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is written as (c, η). Typically, c is represented as the formal sum $c = \sum_{\eta \in X^*} (c, \eta)\, \eta$. For any language $L \subseteq X^*$, its *characteristic series* is defined as $\mathrm{char}(L) = \sum_{\eta \in L} \eta$. The collection of all formal power series over X is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$, while the set of polynomials over X is designated by $\mathbb{R}\langle X\rangle$. Each of these sets forms an associative $\mathbb{R}$-algebra under the catenation (Cauchy) product and a commutative and associative $\mathbb{R}$-algebra under the shuffle product, that is, the $\mathbb{R}$-bilinear mapping $\mathbb{R}^\ell\langle\langle X\rangle\rangle \times \mathbb{R}^\ell\langle\langle X\rangle\rangle \to \mathbb{R}^\ell\langle\langle X\rangle\rangle$ uniquely specified by the shuffle product of two words

$$(x_i\eta) \sqcup\!\sqcup (x_j\xi) = x_i(\eta \sqcup\!\sqcup (x_j\xi)) + x_j((x_i\eta) \sqcup\!\sqcup \xi),$$

where $x_i, x_j \in X,\ \eta, \xi \in X^*$ and with $\eta \sqcup\!\sqcup \emptyset = \eta$ [5,13]. Finally, the *left-shift operator* is defined as $\xi^{-1} : X^* \to \mathbb{R}\langle X\rangle$ such that $\xi^{-1}(\eta) = \eta'$ when $\eta = \xi\eta'$ with $\eta' \in X^*$, and 0 otherwise.

2.1. Rough Paths. The rough path theory presented here is based on the treatment in [6, 11, 12]. A path on $J \triangleq [0, T]$ is a function $U : J \to \mathbb{R}^m$. Let $\mathcal{D}_r \triangleq \{0 < t_1 < \cdots < t_r = T\}$ be a partition of J, and $\mathcal{D}(J)$ denotes the set of all finite partitions of J. The *p-variation* of a path U is

$$\|U\|_{p,J} \triangleq \left(\sup_{\mathcal{D}_r \in \mathcal{D}(J)} \sum_{t_l \in \mathcal{D}_r} |U(t_{l+1}) - U(t_l)|^p \right)^{\frac{1}{p}}.$$

Note that $\|U\|_{p,J} = 0$ only when U is constant, so a norm for the vector space $\mathcal{V}^p(J) \triangleq \{U : J \to \mathbb{R}^d : \|U\|_{p,J} < \infty\}$ is $\|U\|_{\mathcal{V}^p,J} \triangleq \|U\|_{p,J} + \sup_{t\in J} |U(t)|$. It is standard that when $p = 1$ and $\|U\|_{\mathcal{V}^p,J} < \infty$, the path U can act as the integrator for a Stieltjes type integral since U defines a function of bounded variation. In fact, Fliess originally introduced his input-output operators as weighted sums of Stieltjes iterated integrals [5]. Let $\Delta_T \triangleq \{(s,t) \in [0,T] \times [0,T] : 0 \le s \le t \le T\}$, and define the continuous map $\omega : \Delta_T \to \mathbb{R}$ such that

$$\omega(s, \tau) + \omega(\tau, t) \le \omega(s, t)$$

and $\omega(\tau, \tau) = 0$ for $s \le \tau \le t \in J$. Throughout this paper, ω will be called a *ω-function*. In rough path theory an ω-function is referred to as a *control function*. However, that name is not adopted in this paper to avoid the obvious confusion with system theory terminology. It is known that any positive linear combination of ω-functions is an ω-function, and a convex function of an ω-function is also an ω-function [6, Section 1.2]. If ω satisfies $|U(t) - U(s)| \le \omega(s,t)$ for $U \in \mathcal{V}^p(J)$, then it can be verified that $\|U\|_{p,[s,t]} \le \omega(s,t)^{\frac{1}{p}}$ for all $(s,t) \in \Delta_T$. Moreover, ω provides a reparametrization such that U becomes a $1/p$-Hölder continuous path. In particular, a natural ω-function for $U \in \mathcal{V}^1(J)$ is $\omega(s,t) = \|U\|_{1,[s,t]}$ such that $|U(t) - U(s)| \le \|U\|_{1,[s,t]}$. This mapping is additive in that $\|U\|_{1,[s,t]} = \|U\|_{1,[s,\tau]} + \|U\|_{1,[\tau,t]}$, and

$\|U\|_{1,[0,t]}$ as a function of t is continuous. Thus, it has a finite 1-variation. Similarly for $U \in \mathcal{V}^p(J)$, one has $\omega(s,t) = \|U\|^p_{p,[s,t]}$.

Although rough path theory can be developed for paths taking values in a general Banach space, E, here the focus is on the case where $E = \mathbb{R}^{m+1}$ with the usual norm. Moreover, rough paths are normally defined on the tensor space

$$T((E)) = \{a = (a_0, a_1, \ldots) : a_n \in E^{\otimes n}, n \geq 0\} = \bigoplus_{n=0}^{\infty} E^{\otimes n},$$

where $E^{\otimes n} \triangleq \operatorname{span}_{\mathbb{R}}\{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} : i_1, i_2, \ldots, i_n \in I^n, I = \{0, 1, \ldots, n\}\}$ is the $\mathbb{R}$-vector space of all multilinear forms of length n, and $e_0, e_1, \ldots, e_m$ is a basis for E. It is known, however, that the space $T((E))$ is isomorphic to $\mathbb{R}\langle\langle X\rangle\rangle$. Thus, for each $c = (c_0, c_1, \ldots) \in T((E))$ there corresponds a unique $c = \sum_{\eta \in X^*}(c, \eta)\eta \in \mathbb{R}\langle\langle X\rangle\rangle$ and vice-versa. Therefore, the entire development will be done using only $\mathbb{R}\langle\langle X\rangle\rangle$. The sum and product in $T((E))$ are identified with the sum and Cauchy product of formal power series. The space $T((E))$ truncated to order n, denoted by $T^{(n)}(E)$, is isomorphic to the $\mathbb{R}$-vector space of all polynomials of degree n, $\mathbb{R}^{(n)}\langle X\rangle$. Define the $\mathbb{R}$-algebra homomorphism $\pi_n : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}^{(n)}\langle\langle X\rangle\rangle$ such that $c_{(n)} = \pi_n(c)$ is the truncation of c to order n. Then, one can define an associative product of $c, d \in \mathbb{R}^{(n)}\langle\langle X\rangle\rangle$ as $cd \triangleq \pi_n(cd)$.

Let $U \in \mathcal{V}^1(J)$. For every $\eta \in X^*$, denote by $E_\eta[U]$ the iterated Stieltjes integral of U defined inductively by setting $E_\emptyset[U] = 1$ and

$$E_{x_i\eta'}[U](t_2, t_1) = \int_{t_1}^{t_2} dU_i(\tau) E_{\eta'}[U](\tau, t_1),$$

where $x_i \in X$ and $\eta' \in X^*$.

Definition 2.1. *A* **multiplicative functional of degree** $n \geq 1$ *is a continuous map* $P_n[U] : \Delta_T \to \mathbb{R}^{(n)}\langle X\rangle$ *defined as* $P_n[U](t_2, t_1) \triangleq \sum_{\eta \in X^k, k \leq n} E_\eta[U](t_2, t_1)\, \eta$ *that satisfies*

$$P_n[U](t_2, t_1) = \pi_n(P_n[U](t_2, \tau) P_n[U](\tau, t_1))$$

for $t_1 \leq \tau \leq t_2$ *and is referred to as* **Chen's identity**.

A notion of p-variation for $P_n[U]$ is given next.

Definition 2.2. *The map* $P_n[U] : \Delta_T \to \mathbb{R}^{(n)}\langle X\rangle$ *is said to have* **finite total *p*-variation** *if*

$$\sup_{\mathcal{D}_r \in \mathcal{D}(J)} \sum_{t_l \in \mathcal{D}_r} |E_\eta[U](t_l, t_{l-1})|^{\frac{p}{|\eta|}} < \infty$$

when $|\eta| \leq n$.

Lemma 2.3. *[12, Proposition 3.3.2] Let* $P_n[U]$ *be a multiplicative functional of order* n*. If* $P_n[U]$ *has finite total* p*-variation then for* $(s,t) \in \Delta_T$

$$\omega(s,t) = \sum_{j=1}^{n} \sum_{\eta \in X^j} \sup_{\mathcal{D}_r \in \mathcal{D}(J)} \sum_{t_l \in \mathcal{D}_r} |E_\eta[U](t_l, t_{l-1})|^{\frac{p}{|\eta|}}$$

is a ω*-function, and*

$$\|E_\eta[U](t,s)\|_{\mathcal{V}^p(J)} \leq \omega(s,t)^{\frac{|\eta|}{p}}.$$

For paths with p-variation greater than 1, the idea is to make sense of the differential dU. It is known for ordinary differential equations that by taking the limit of the first order increment processes $E_{x_i}[U](t,s) = \{U_i(t) - U_i(s) : 0 \leq s \leq t\}$, $i = 0, 1, \ldots, m$, as the partition size goes to zero, one can obtain a useful characterization of dU without ever computing them explicitly. Therefore, one may regard the whole collection of first order increments as the *differential* dU when $p = 1$. Employing Chen's identity,

$$E_\eta[U](t_2,t_1) = \lim_{\substack{r\to\infty \\ |\mathcal{D}_r|\to 0}} \sum_{\substack{t_l\in\mathcal{D}_r \\ \eta=\eta_1\eta_2}} E_{\eta_1}[U](t_l,t_{l-1})E_{\eta_2}[U](t_{l-1},t_1), \tag{2.1}$$

where $|\mathcal{D}_r|$ denotes the partition size. So all higher order increments are well defined in terms of the first order increment process, i.e.,

$$\sup_{\mathcal{D}_r\in\mathcal{D}(J)} \sum_{t_l\in\mathcal{D}_r} |E_\eta[U](t_l,t_{l-1})|^{\frac{p}{|\eta|}} < \infty, \ \ \forall \eta \in X^*, \tag{2.2}$$

where $p = 1$. To describe dU when $p > 1$, one needs higher order increments as well. Observe that (2.2) is not satisfied even for $|\eta| = 1$ and $p = 2$, but the increments do satisfy the weaker condition

$$\sup_{\mathcal{D}_r\in\mathcal{D}(J)} \sum_{t_l\in\mathcal{D}_r} |U(t_l) - U(t_{l-1})|^{p'} < \infty$$

for any $p' \geq p$. Thus, the iterated path integrals defined in (2.1) do converge in a p-variation metric. Moreover, Lyons showed that if, in addition, U satisfies Chen's identity, then $E_\eta[U]$ is uniquely determined when $\eta \in X^k$, $k \geq \lfloor p \rfloor + 1$ by $\{E_\eta[U] : |\eta| = 1, 2, \ldots \lfloor p \rfloor\}$.

THEOREM 2.4. *[11, 12] Let $p \geq 1$, and let $P_n[U]$ be a multiplicative functional of order n with finite p-variation so that*

$$\|E_\eta[U](t,s)\|_{p,[s,t]} \leq \omega(s,t)^{\frac{|\eta|}{p}}$$

for $|\eta| \leq n$ and some ω-function. If $n \geq \lfloor p \rfloor$ then $P_n[U]$ can be extended uniquely to a finite p-variation multiplicative functional $P[U] \in \mathbb{R}\langle\langle X \rangle\rangle$. Moreover, there exist a $K > 0$ such that for every $\eta \in X^*$ it follows that

$$\|E_\eta[U]\|_{p,[s,t]} \leq \frac{K\omega(s,t)^{\frac{|\eta|}{p}}}{\left(\frac{|\eta|}{p}\right)!}, \ \ \forall (s,t) \in \Delta_T. \tag{2.3}$$

This result is known as the *extension theorem*, and it is the first fundamental theorem of rough path theory. In this sense, p indicates how many iterated path integrals (or higher order increments) are needed in order to have a well-posed integration theory. A crucial result used in the proof of the extension theorem is the so-called *neo-classical inequality* given in the next lemma.

LEMMA 2.5. *[9, 11] For any $p \geq 1$, $n \in \mathbb{N}$ and $t_1, t_2 \geq 0$,*

$$\frac{1}{p}\sum_{i=0}^{n} \frac{t_1^{\frac{i}{p}}}{\left(\frac{i}{p}\right)!} \frac{t_2^{\frac{n-i}{p}}}{\left(\frac{n-i}{p}\right)!} \leq \frac{(t_1+t_2)^{\frac{n}{p}}}{\left(\frac{n}{p}\right)!}.$$

Let $U = (U_1, \ldots, U_m) \in \mathcal{V}^p(J)$ and $U_0(t) \triangleq t$. The *signature* (or *Chen series*) associated with U is an element of $\mathbb{R}\langle\langle X\rangle\rangle$ defined as

$$P[U](t_2, t_1) = \sum_{\eta \in X^*} E_\eta[U](t_2, t_1)\, \eta.$$

Here all iterated integrals are in the Lyons sense. In particular, if

$$P[U](t_2, t_1) = P[U](t_2, \tau) P[U](\tau, t_1), \quad t_1 \leq \tau \leq t_2 \tag{2.4}$$

then $P[U]$ is said to be *multiplicative*. Identity (2.4) was originally given as a theorem by Chen in [1] when $p = 1$. However, in rough path theory it is a purely algebraic property that a path with arbitrary p-variation must satisfy in order to behave properly as an integrator.

Definition 2.6. *A* ***p*-rough path** *is a multiplicative functional of degree $\lfloor p \rfloor$ in $\mathbb{R}^m$ having finite p-variation. The space of p-rough paths is denoted by $\Omega_p(\mathbb{R}^m)$.*

Thus, a p-rough path is a continuous mapping from Δ_T to $\mathbb{R}^{(\lfloor p \rfloor)}\langle X\rangle$, which is a multiplicative functional of degree $\lfloor p \rfloor$ and has finite total p-variation.

An important class of rough paths is the subset of all rough paths that are limits of 1-rough paths in the p-variation metric. They are known as *geometric rough paths*, and the set of all such paths is denoted by $G\Omega_p(\mathbb{R}^m)$. This set constitutes the input class for Fliess signature operators as defined later in Section 3. The goal is then to describe a Fliess signature operator that maps a p-rough path to another p-rough path, thus allowing one to drive a second Fliess signature operator with this signal.

2.2. Fliess Operators. A Fliess operator is formally defined in terms of the signature of a path as follows.

Definition 2.7. *Let $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $U \in G\Omega_p(\mathbb{R}^m)$. The corresponding* **Fliess operator** *driven by U is*

$$F_c[U] = (c, P[U]) \triangleq \sum_{\eta \in X^*} (c, \eta)(P[U], \eta) = \sum_{\eta \in X^*} (c, \eta) E_\eta[U].$$

Note in this setting that the output path generated by a Fliess operator is $Y = \int dt\, F_c(U) = F_{x_0 c}[U]$. The integral only coincides with Lebesgue integration when $p = 1$. This is analogous to generating a Wiener process by integrating white Gaussian noise. This reinforces the fact that a Fliess operator output is obtained as an approximation of smooth signals. A condition must be imposed on c in order to give some notion of convergence for a Fliess operator.

Definition 2.8. *A series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ is called* **Gevrey of degree *r*** *if*

$$|(c, \eta)| \leq K M^{|\eta|} (|\eta|!)^r, \quad \eta \in X^*$$

for some $K, M > 0$ and $r \geq 0$. The set of all such series is denoted by $\mathbb{R}^\ell_{G(r)}\langle\langle X\rangle\rangle$.

Theorem 2.9. *Let $c \in \mathbb{R}^\ell_{G(1/p')}\langle\langle X\rangle\rangle$. If $U \in G\Omega_p(\mathbb{R}^m)$ with $p < p'$ and $Y = F_{x_0 c}[U]$, then $\|F_c[U]\|_{\mathcal{V}^p(J)} < \infty$. Under the same assumptions, if $p = p'$ and $M(m + 1)\, \omega(0,t)^{1/p} < 1$ then $\|F_c[U]\|_{\mathcal{V}^p(J)} < \infty$.*

Thus, given an input in $G\Omega_p(\mathbb{R}^m)$, each output component generated by F_c is in $\mathcal{V}^p$. But this means that a cascade interconnection is still not well-posed since being in $\mathcal{V}^p$ does not imply that the output is a p-rough path.

When F_c and F_d with $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ are interconnected in a cascade fashion, the composite system always has at least a formal Fliess operator representation in terms of the composition product [7]. It is convenient to first describe this product using a family of mappings

$$D_{x_i} : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}\langle\langle X\rangle\rangle : e \mapsto x_0(d_i \sqcup\!\sqcup\, e),$$

where $i = 0, 1, \ldots, m$ and $d_0 := 1$. Let $D_\emptyset$ be the identity map on $\mathbb{R}\langle\langle X\rangle\rangle$. Such maps can be composed in an obvious way so that $D_{x_i x_j} := D_{x_i} D_{x_j}$ provides an $\mathbb{R}$-algebra.

Definition 2.10. *[7] The* **composition product** *of $\eta \in X^*$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ is*

$$\eta \circ d \triangleq (x_{i_k} x_{i_{k-1}} \cdots x_{i_1}) \circ d = D_{x_{i_k}} D_{x_{i_{k-1}}} \cdots D_{x_{i_1}}(1) = D_\eta(1).$$

For any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$,

$$c \circ d \triangleq \sum_{\eta \in X^*} (c, \eta)\, \eta \circ d.$$

The composition product is associative and satisfies $F_c \circ F_d = F_{c \circ d}$.

3. Fliess Signature Operators. The goal of this section is to explicitly describe the mapping between the input signature and the output signature of a Fliess operator. To achieve this objective, a device known as a *transduction* is employed.

Definition 3.1. *[4, 10] Let X and W be two alphabets. Any $\mathbb{R}$-linear mapping $t : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}\langle\langle W\rangle\rangle$ is called a* **transduction**. *It is completely specified by*

$$t(\eta) = \sum_{\xi \in W^*} (t(\eta), \xi)\xi, \quad \eta \in X^*.$$

With any t one can canonically associate a series in $\mathbb{R}\langle\langle X \otimes W\rangle\rangle$, namely

$$\hat{t} = \sum_{\eta \in X^*} \eta \otimes t(\eta) = \sum_{\eta \in X^*, \xi \in W^*} (t(\eta), \xi)\, \eta \otimes \xi.$$

Given the fact that $\hat{t}$ is still a formal power series in the usual sense, transductions have a well-defined notion of Gevrey degree.

Definition 3.2. *Let $X = \{x_0, x_1, \ldots, x_m\}$ and $W = \{w_0, w_1, \ldots, w_\ell\}$. A transduction $t_c : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}\langle\langle W\rangle\rangle$ is said to be* **associated** *with $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ if it can be written as*

$$t_c(\eta) = \sum_{\xi \in W^*} (\xi \circ c, \eta)\, \xi, \quad \eta \in X^*.$$

The corresponding series $\hat{t}_c \in \mathbb{R}\langle\langle X \otimes W\rangle\rangle$ is then $\hat{t}_c = \sum_{\xi \in W^*} (\xi \circ c) \otimes \xi$.

Definition 3.3. *For $c \in \mathbb{R}^m\langle\langle X\rangle\rangle$, define the* **Fliess signature operator** *as*

$$\begin{aligned} S_c \;&:\; \mathbb{R}\langle\langle X\rangle\rangle \;\to\; \mathbb{R}\langle\langle W\rangle\rangle \\ &:\; P[U] \;\mapsto\; P[Y] = \sum_{\eta \in X^*} t_c(\eta)(P[U], \eta). \end{aligned}$$

Observe then that, as expected,

$$S_c(P[U]) = \sum_{\eta \in X^*, \xi \in W^*} \xi\, (\xi \circ c, \eta) E_\eta[U] = \sum_{\xi \in W^*} \xi\, E_\xi[F_c[U]] = \sum_{\xi \in W^*} \xi\, E_\xi[Y] = P[Y],$$

where $Y = F_{x_0 c}[U]$. It is also clear from the above calculation that

$$E_\xi[Y] = \sum_{\eta \in X^*} (\xi \circ c, \eta) E_\eta[U], \ \ \xi \in W^*. \tag{3.1}$$

Fliess signature operators as defined above are purely algebraic objects. It is necessary therefore to show that $S_c(P[U])$ is a proper p-rough path when U is a p-rough path. But first one needs to make sense of $E_\xi[Y]$, $\xi \in W^*$, by showing that the series (3.1) converges. Then, it will be proved that $P[Y] = S_c(P[U])$ is a multiplicative functional of order $\lfloor p \rfloor$. These two steps provide the means to show that $E_\xi[Y]$, $\xi \in W^*$, satisfies (2.3). In order to show that the series defining $E_\xi[Y]$ is well-defined, a characterization of the Gevrey degree of the series $\xi \circ c$ is given in the next theorem.

THEOREM 3.4. *Let $c \in \mathbb{R}^\ell_{G(1/p)}\langle\langle X \rangle\rangle$ with growth constants $K_c, M_c > 0$ and $p \geq 1$. Then for any $\xi \in W^*$ it follows that $\xi \circ c \in \mathbb{R}_{G(1/p)}\langle\langle X \rangle\rangle$. Specifically,*

$$|(\xi \circ c, \theta)| \leq (K_c^k M_c^{-|\xi|})(2^k M_c)^{|\theta|}(|\theta|!)^{\frac{1}{p}},$$

for all $\theta \in X^*$ such that $|\theta| \geq |\xi|$ and where $k = |\xi| - |\xi|_{x_0}$. Otherwise, $(\xi \circ c, \theta) = 0$.

A straightforward consequence of this theorem is the following corollary, which says that $E_\xi[Y]$, $\xi \in W^*$, is well-defined.

COROLLARY 3.5. *Let $c \in \mathbb{R}^\ell_{G(1/p')}\langle\langle X \rangle\rangle$. If $U \in G\Omega_p(\mathbb{R}^m)$ with $p \leq p'$ and $Y = F_{x_0 c}[U]$ then*

$$\|E_\xi[Y]\|_{\mathcal{V}^p(J)} < \infty, \ \ \xi \in W^*.$$

The next lemma is key to showing that $P[Y]$ is multiplicative of order $\lfloor p \rfloor$.

LEMMA 3.6. *Let $c \in \mathbb{R}^\ell_{G(1/p')}\langle\langle X \rangle\rangle$. If $U \in G\Omega_p(\mathbb{R}^m)$ with $p \leq p'$ and $Y = F_{x_0 c}[U]$ then*

$$E_\xi[Y](t_2, t_1) = \sum_{\xi = \xi_1 \xi_2} E_{\xi_1}[Y](t_2, \tau) E_{\xi_2}[Y](\tau, t_1),$$

where $\xi \in W^*$ and $t_1 \leq \tau \leq t_2$.

THEOREM 3.7. *Let $c \in \mathbb{R}^\ell_{G(1/p')}\langle\langle X \rangle\rangle$. If $U \in G\Omega_p(\mathbb{R}^m)$ with $p \leq p'$ and $Y = F_{x_0 c}[U]$ then $P_{\lfloor p \rfloor}[Y]$ is a multiplicative functional of order $\lfloor p \rfloor$.*

The next lemma is needed in order to show that $E_\xi[Y]$ satisfies (2.3).

LEMMA 3.8. *Let $c \in \mathbb{R}^\ell_{G(1/p')}\langle\langle X \rangle\rangle$. If $U \in G\Omega_p(\mathbb{R}^m)$ with $p \leq p'$ and $Y = F_{x_0 c}[U]$ then there exist an ω-function such that*

$$\|E_\xi[Y]\|_{p,[t_1,t_2]} \leq \omega(t_1, t_2)^{\frac{|\xi|}{p}}, \ \ \xi \in W^*, \ \ |\xi| \leq \lfloor p \rfloor.$$

LEMMA 3.9. *Let $c \in \mathbb{R}^\ell_{G(1/p')}\langle\langle X \rangle\rangle$. If $U \in G\Omega_p(\mathbb{R}^m)$ with $p \leq p'$ and $Y = F_{x_0 c}[U]$ then*

$$\|E_\xi[Y]\|_{p,[t_1,t_2]} \leq \frac{\omega(t_1, t_2)^{\frac{|\xi|}{p}}}{\left(\frac{|\xi|}{p}\right)!}, \ \ 0 \leq t_1 \leq t_2, \ \ \xi \in W^*.$$

It is thus trivial using Theorem 3.7 and Lemma 3.9 to conclude that $P[Y]$ is indeed a p-rough path. Furthermore, the next lemma gives a bound for the summation of iterated integrals of the same order with arbitrary p-variation inputs.

LEMMA 3.10. *Let $c \in \mathbb{R}^{\ell}_{G(1/p')}\langle\langle X\rangle\rangle$. If $U \in G\Omega_p(\mathbb{R}^m)$ with $p \le p'$ and $Y = F_{x_0 c}[U]$ then for any $k \ge 1$*

$$\left\| \sum_{\xi \in W^k} E_\xi[Y] \right\|_{p,[t_1,t_2]} \le (m+1)^{k-1} \frac{\omega(t_1,t_2)^{\frac{k}{p}}}{k!}, \quad 0 \le t_1 \le t_2.$$

Finally, the main result of the section is given below.

THEOREM 3.11. *Let $c \in \mathbb{R}^{\ell}_{G(1/p')}\langle\langle X\rangle\rangle$. If $U \in G\Omega_p(\mathbb{R}^m)$ with $p \le p'$ then the Fliess signature operator S_c maps $G\Omega_p(\mathbb{R}^m)$ into $G\Omega_p(\mathbb{R}^\ell)$.*

4. Cascade Interconnections with Rough Path Inputs. In light of Theorem 3.11, the output of a Fliess signature operator driven by an p-rough path input U can be fed into a second Fliess signature operator as long as its generating series is in $\mathbb{R}^m_{G(1/p)}\langle\langle X\rangle\rangle$. The cascade is best described algebraically by a composition of transductions.

DEFINITION 4.1. *For $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$, the* **composition of transductions** *t_c and t_d is*

$$t_c \circ t_d = \sum_{\eta \in X^*, \xi \in W^*} (\xi \circ c, \eta)\, \eta \circ d \otimes \xi.$$

LEMMA 4.2. *For transductions $\hat{t}_c = \sum_{\xi \in W^*} (\xi \circ c) \otimes \xi$ and $\hat{t}_d = \sum_{\eta \in X^*} (\eta \circ d) \otimes \eta$ it follows that $t_c \circ t_d = t_{c \circ d}$.*

Using the composition of transductions, the following theorem is easily proved.

THEOREM 4.3. *The cascade of two Fliess signature operators S_c and S_d is described by $S_c \circ S_d = S_{c \circ d}$.*

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