SMOOTH UNIVERSAL INPUTS FOR SMOOTH SYSTEMS: A FORMAL POWER SERIES APPROACH*

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Abstract. The goal is to prove the existence of smooth universal inputs for smooth systems in the context of formal power series and Fliess operators. That is, nonlinear operators described by functional series which are indexed by words over a noncommutative alphabet. Their generating series are therefore specified in terms of noncommutative formal power series. The idea is to provide a more direct and elementary existence argument than is currently available, thereby making this result more accessible to researchers with different backgrounds. In addition, this approach will illustrate how the behavior of smooth systems can be described by formal power series, which have traditionally been used for analytic systems.

Key words. nonlinear control systems, observability, formal power series

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1. Introduction.

A useful concept in observability theory and parameter identification is the property of *state distinguishability.* Two states p and q of a given state space system are said to be distinguishable if there exists an input u which produces distinct outputs when the system is initialized, respectively, at p and q. In this case, u is said to distinguish p and q. An input u is a *universal input* if u distinguishes all distinguishable pairs p and q. Aside from its intrinsic appeal as a theoretical concept, the existence of universal inputs has practical applications for identification in systems biology [12] and for path planning [7, 8].

The first results on the existence of universal inputs appeared in 1977 for bilinear systems [4]. Analogous results were established shortly thereafter in [6] for discrete-time systems and analytic continuous-time systems with compact state spaces. This analysis was extended to analytic continuous-time systems without the compactness condition in [10]. The main results in [10] were then proved in [13] by alternative methods in the context of input-output equations. In [9], the existing theorems on universal inputs for analytic systems were extended to show that there are smooth inputs (that is, C^∞ functions) that are universal for *all* finite dimensional analytic systems. A counterexample was provided to show that there does not exist such an analytic universal input for *all* finite dimensional analytic systems. The same example can also be used to show that there does not exist a smooth universal input for *all* finite dimensional smooth systems.

The existence of universal inputs for a given smooth system was established in the seminal paper [1]. The main focus of [1] was on smooth stabilization. Using sophisticated methods from differential geometry, it was shown that for a generic feedback function $u(x)$, a certain controllability property holds for the linearized systems around each regular trajectory. This led to important results concerning smooth stabilization. The existence of universal inputs was obtained as a corollary to these main results. In fact, the primary result in this regard was in the context of observation spaces, which implies the existence of smooth universal inputs.

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The main goal of this paper is to prove the existence of smooth universal inputs for smooth systems in the context of formal power series and Fliess operators. That is, nonlinear operators described by functional series which are indexed by words over a noncommutative alphabet [2]. Their generating series are therefore specified in terms of noncommutative formal power series. The idea is to provide a more direct and elementary existence argument, thereby making this result more accessible to researchers with different backgrounds. In addition, this approach will illustrate how the behavior of smooth systems can be described by formal power series (which have traditionally been used for analytic systems). The approach is based on methods for *formal* Fliess operators in [5].

The paper is organized as follows. In the next section some mathematical preliminaries on Fliess operators and state distinguishability are briefly reviewed. The concept of zero-distinguishability is introduced in the subsequent section. It is utilized in Section 4 in order to define a set of generic input jets in terms of formal Fliess operators. In the next section, this concept is used to show the existence of smooth universal inputs. Given the space limitations, some of the proof details will be deferred to a future publication. The conclusions of the paper are summarized in the final section.

2. Preliminaries.

2.1. Formal Power Series and Fliess Operators.

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. The *length* of η, $|\eta|$, is the number of letters in η, while $|\eta|_{x_i}$ is the number of times the letter x_i appears in η. The set of all words with length k is denoted by X^k. The set of all words including the empty word, $\emptyset$, is designated by X^*. It forms a monoid under catenation. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is written as (c, η) and called the *coefficient* of η in c. The collection of all formal power series over X is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. It forms an associative $\mathbb{R}$-algebra under the catenation product and a commutative and associative $\mathbb{R}$-algebra under the shuffle product [2].

One can formally associate with any series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} :\ 1 \leq i \leq m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L^m_\mathfrak{p}[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm and $B^m_\mathfrak{p}(R)[t_0, t_1] := \{u \in L^m_\mathfrak{p}[t_0, t_1] : \|u\|_\mathfrak{p} \leq R\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L^m_1[t_0, t_1]$. Define iteratively for each $\eta \in X^*$ the map $E_\eta : L^m_1[t_0, t_1] \to \mathcal{C}[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The input-output operator corresponding to c is the *Fliess operator*

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0) \tag{2.1}$$

[2].

The cascade connection of two Fliess operators is known to always yield an input-output system having a Fliess operator representation (see the citations in [5]). To describe its generating series explicitly, let $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ and define the family of mappings

$$D_{x_i} : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}\langle\langle X\rangle\rangle : e \mapsto x_0(d_i \sqcup\!\sqcup\, e),$$

where $i = 0, 1, \ldots, m$ and $d_0 := 1$. Assume $D_\emptyset$ is the identity map on $\mathbb{R}\langle\langle X\rangle\rangle$. Such maps can be composed in the obvious way so that $D_{x_i x_j} := D_{x_i} D_{x_j}$. The *composition product* of a word $\eta \in X^*$ and a series $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ is defined as

$$(\underbrace{x_{i_k} x_{i_{k-1}} \cdots x_{i_1}}_{\eta}) \circ d = D_{x_{i_k}} D_{x_{i_{k-1}}} \cdots D_{x_{i_1}}(1) = D_\eta(1).$$

For any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ the definition is extended linearly as

$$c \circ d = \sum_{\eta \in X^*} (c, \eta)\, \eta \circ d.$$

In which case, for any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$, the identity $F_c \circ F_d = F_{c \circ d}$ is satisfied.

Convergence of the series (2.1) defining F_c can be guaranteed by introducing certain growth restrictions on the coefficients of c. Otherwise, F_c can be interpreted only as a mapping between the generating series for a formal input, $c_u \in \mathbb{R}^m[[X_0]]$, and the generating series for the formal output, $c_y \in \mathbb{R}^\ell[[X_0]]$, where $X_0 = \{x_0\}$. Such a mapping can be described explicitly using the composition product. Namely, the class of *formal Fliess operators* is defined as the set of mappings

$$\mathscr{F} := \left\{ c\circ : \mathbb{R}^m[[X_0]] \to \mathbb{R}^\ell[[X_0]] : c_u \mapsto c_y = c \circ c_u,\ c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle \right\}. \tag{2.2}$$

It was shown in [5] that the generating series, c, of a formal Fliess operator is unique.

2.2. State Distinguishability. Consider a smooth system

$$\begin{aligned} \dot{x} &= g_0(x) + \sum_{i=1}^m g_i(x) u_i, \\ y &= h(x), \end{aligned} \tag{2.3}$$

where for each t, $x(t) \in \mathcal{M}$, which is a smooth, second countable manifold of dimension n, the output map $h : \mathcal{M} \to \mathbb{R}$ is a smooth function, and $g_0, g_1, \ldots, g_m$ are smooth vector fields defined on $\mathcal{M}$. Inputs of the system are measurable essentially bounded maps $u : [0, T] \to \mathbb{R}^m$ defined on $[0, T]$ for some $T > 0$. Let $\varphi(t, x_0, u)$ denote the trajectory of (2.3) corresponding to an input u defined on some interval $[0, b)$ for $b > 0$ and an initial state x_0, and let $y(t, x_0, u) = h(\varphi(t, x_0, u))$.

Let $p, q \in \mathcal{M}$ and take an input u. Then p and q are said to be *distinguishable by* u, denoted by $p \not\sim_u q$, if $y(t, p, u) \neq y(t, q, u)$ on the common interval where both functions are defined. Two states p and q are *distinguishable*, denoted by $p \not\sim q$, if $p \not\sim_u q$ for some input u. Finally, p and q are called *indistinguishable*, denoted by $p \sim q$, when p and q cannot be distinguished by any input u. The following lemma is easy to prove.

LEMMA 2.1. *Let $p, q \in \mathcal{M}$. Then $p \not\sim q$ if and only if there exists a polynomial input u such that $p \not\sim_u q$.*

An input function u is called a *universal input* if $p \not\sim_u q$ whenever $p \not\sim q$. The goal is to show that there always exists a smooth universal input for the smooth system (2.3). For this purpose, consider an augmented version of (2.3):

$$\begin{aligned} \dot{x} &= g_0(x) + \sum_{i=1}^{m} g_i(x)u_i, \\ \dot{z} &= g_0(z) + \sum_{i=1}^{m} g_i(z)u_i, \\ y_o &= h(x) - h(z). \end{aligned} \tag{2.4}$$

It is evident that (2.3) admits a universal input if and only if there exists some input u such that for (2.4), $(p,q) \not\sim_u (p,p)$ whenever $(p,q) \not\sim (p,p)$. This motivates the study of *zero-distinguishability* as described next.

3. Zero-Distinguishability. Consider system (2.3) and assume that there exists some $p_o \in \mathcal{M}$ such that $y(t,p_o,u) = 0$ for all $t \geq 0$ and all input functions u. An input u is called a *zero-distinguishable universal input* if $p \not\sim_u p_o$ for every $p \not\sim p_o$. It can be seen that u is a zero-distinguishable universal input for (2.4) if and only if u is a universal input for (2.3). For any $\mu = (\mu_0, \mu_1, \ldots) \in \mathbb{R}^{m,\infty}$ (which is taken to be $\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \cdots$), define for each $i \geq 1$,

$$\psi_i(p,\mu) = \left.\frac{d^i}{dt^i}\right|_{t=0^+} h(\varphi(t,p,u)),$$

where u is any C^∞ input with initial values $u^{(j)}(0) = \mu_j$, $j \geq 0$. Observe that the functions $\psi_i(p,\mu)$ can be expressed as a polynomial in the variables $(\mu_0, \mu_1, \ldots, \mu_{i-1})$. Define $\psi_0(p,\mu) = h(p)$.

For system (2.3) and each $p \in \mathcal{M}$, define the series

$$c_p = \sum_{\eta \in X^*} (c_p, \eta)\eta,$$

where

$$(c_p, x_{i_1} x_{i_2} \cdots x_{i_k}) = L_{g_{i_k}} \cdots L_{g_{i_2}} L_{g_{i_1}} h(p),$$

and the *Lie derivative* is given by $L_{g_i} : h \mapsto (\partial h / \partial z) \cdot g_i$. If (2.3) is an analytic system, then the series c_p is a convergent series for each $p \in \mathcal{M}$ (see Lemma 4.2 in [11]). However, for a smooth system (2.3), the series c_p is not necessarily convergent. By Theorem 7 of [5], if $c_p \neq 0$, then $\psi_i(p,\mu) \neq 0$ for some i and $(\mu_0, \mu_1, \ldots, \mu_{i-1})$. It then follows that $y(\cdot, p, u) \neq 0$, where u is any input with the initial values $u^{(j)}(0) = \mu_j$ for $j \leq i-1$. This implies that $p \not\sim p_o$. On the other hand, if $c_p = 0$, then it is immediate that

$$L_{g_{i_k}} \cdots L_{g_{i_2}} L_{g_{i_1}} h(p) = 0$$

for all choices of $i_1, i_2, \ldots i_k$ and all $k \geq 0$. If system (2.3) is analytic, this then implies that $y(\cdot, p, u) = 0$ for all input u, i.e., $p \sim p_o$. However, this property may fail for a smooth system.

EXAMPLE 3.1. Consider the smooth system

$$\dot{x} = xu, \quad y = h(x),$$

where $h(x) = \mathrm{e}^{-\frac{1}{(x-1)^2}}$ if $x > 1$, and $h(x) = 0$ if $x \leq 1$. Clearly, $c_{p_o} = 0$ when $p_o = 0$ and $y(t, 0, u) = 0$, $t \geq 0$ for all u. In addition, $L^i_{g_1} h(1) = 0$ for all $i \geq 0$, and hence $c_p = 0$ when $p = 1$. But for $u(t) = \frac{1}{1+t}$, it holds that $\varphi(t, 1, u) = 1 + t$, and hence, $y(t, 1, u) = \mathrm{e}^{-\frac{1}{t^2}}$, $t > 0$. Thus, $p \not\sim_u p_o$ even though $c_p = c_{p_o} = 0$.

In this work, the following *regularity condition* will be imposed on (2.3) when considering the zero-distinguishability property.

Assumption A_0. For each point $p \in \mathcal{M}$ and any polynomial input u such that

$$\left.\frac{d^k}{dt^k}\right|_{t=0^+} h(\varphi(t, p, u)) = 0, \quad k \geq 0, \tag{3.1}$$

it follows that $y(t, p, u) \equiv 0$.

4. Zero-Distinguishable Universal Input Jets. For any $\mu \in \mathbb{R}^{m,\infty}$, define

$$c_u(\mu) = \sum_{i=0}^{\infty} \mu_i x_0^i.$$

In light of (2.2), it follows for system (2.3) that $\psi_i(p, \mu) = (c_p \circ c_u(\mu),\ x_0^i)$, $i \geq 0$. Assume $\mathbb{R}^{m,\infty}$ is endowed with the product topology whose basis of open sets consists of all sets of the form $U_0 \times U_1 \times U_2 \times \cdots$, where each U_i is an open subset of $\mathbb{R}^m$, and $U_i = \mathbb{R}^m$ for all but finitely many i. With this topology, $\mathbb{R}^{m,\infty}$ is a Baire space, and thus, for a sequence $\{\mathcal{U}_i\}$ of open dense subsets of $\mathbb{R}^{m,\infty}$, $\bigcap_i \mathcal{U}_i$ is a dense subset of $\mathbb{R}^{m,\infty}$. A subset $\mathcal{U}$ of $\mathbb{R}^{m,\infty}$ is called a *generic set* if $\mathcal{U}$ is a countable intersection of open dense subsets of $\mathbb{R}^{m,\infty}$.

Consider the subset $\mathcal{J}$ of $\mathbb{R}^{m,\infty}$ defined by

$$\mathcal{J} = \{\mu \in \mathbb{R}^{m,\infty} :\ c_p \circ c_u(\mu) \neq 0 \quad \forall\, p \in \mathcal{M}_1\},$$

where $\mathcal{M}_1$ is the subset of $\mathcal{M}$ consisting of all states that are distinguishable from p_o, that is,

$$\mathcal{M}_1 = \{p \in \mathcal{M} :\ c_p \neq 0\}.$$

If $c_p = 0$, then (3.1) holds for all k and all smooth input u. By Assumption A_0, $p \sim p_o$. On the other hand, suppose $p \sim p_o$. Then $y(t, p, u) \equiv 0$ (on its interval of definition) for all smooth inputs u, and hence, $c_p \circ c_u(\mu) = 0$ for all $\mu \in \mathbb{R}^{m,\infty}$. By the uniqueness property, $c_p = 0$. Therefore,

$$\mathcal{M}_1 = \{p \in \mathcal{M} :\ p \not\sim p_o\}.$$

It is evident that $\mu \in \mathcal{J}$ if and only if for each $p \not\sim p_o$, there is some i such that $\psi_i(p, \mu) \neq 0$. Moreover, for each $\mu \in \mathcal{J}$, if $u \in \mathbb{C}^\infty$ is a smooth function with $u^{(i)}(0) = \mu_i$ for $i \geq 0$, then u is a zero-distinguishability universal input. By Borel's Theorem, there is always such an input function for each $\mu \in \mathcal{J}$. The elements in $\mathcal{J}$ will be referred to as *zero-distinguishable universal input jets.*

Theorem 4.1. *For system (2.3), $\mathcal{J}$ is a generic subset of* $\mathbb{R}^{m,\infty}$.

Below some technical results required for the proof of this theorem are presented. For each subset D of $\mathcal{M}_1$, define

$$\mathcal{J}_D = \{\mu \in \mathbb{R}^{m,\infty} :\ c_p \circ c_u(\mu) \neq 0 \quad \forall\, p \in D\}.$$

PROPOSITION 4.2. *If D is a compact subset of $\mathcal{M}_1$, then $\mathcal{J}_D$ is open and dense in $\mathbb{R}^{m,\infty}$.*

The next result is a consequence of the fact that $\mathcal{M}$ is second countable.

LEMMA 4.3. *There exist a sequence of compact subsets of $\mathcal{M}_1$, denoted by $\{D_i\}$, such that $\mathcal{M}_1 = \bigcup_{i=1}^{\infty} D_i$.*

Observe that Theorem 4.1 follows from Proposition 4.2 in conjunction with Lemma 4.3. That is, if $\mathcal{M}_1 = \bigcup_i D_i$, where D_i is compact, it then holds that $\mathcal{J} = \bigcap_i \mathcal{J}_{D_i}$, and each $\mathcal{J}_{D_i}$ is open and dense. It will then follow that $\mathcal{J}$ is generic. Thus, to prove Theorem 4.1, it is sufficient to prove Proposition 4.2.

For each given $\mu = (\mu_0, \mu_1, \ldots) \in \mathbb{R}^{m,\infty}$, $(c_p \circ c_u(\mu),\ x_0^i)$ depends only on $(\mu_0, \ldots, \mu_{i-1})$. For each $\nu = (\nu_0, \ldots, \nu_{k-1}) \in \mathbb{R}^{mk}$, let $[c_p \circ c_u(\nu)]_k$ denote the vector

$$\big((c_p \circ c_u(\mu), \emptyset),\ (c_p \circ c_u(\mu), x_0), \ldots,\ (c_p \circ c_u(\mu), x_0^k)\big),$$

where $\mu \in \mathbb{R}^{m,\infty}$ is any extension of ν in the sense that $\mu_i = \nu_i$ for $0 \le i \le k-1$. For a subset $D \subseteq \mathcal{M}$ and $k \ge 0$, let

$$\mathcal{J}_D^k = \{\nu \in \mathbb{R}^{mk} :\ [c_p \circ c_u(\nu)]_k \neq 0 \quad \forall\, p \in D\}$$

(which is not assumed to be nonempty a priori). Let $\nu \in \mathbb{R}^{mk}$. Then μ is a *finite* extension of ν if $\mu \in \mathbb{R}^{mj}$ for some $j \ge l$ such that $\mu_i = \nu_i$ for $0 \le i \le k-1$. For a compact subset D of $\mathcal{M}$, consider the following.

LEMMA 4.4. *Assume that $D \subseteq \mathcal{M}$ is compact and that $\mathcal{J}_D^s \neq \emptyset$ for some $s \ge 1$. Then, for any $k \ge 1$ and any $\nu \in \mathbb{R}^{mk}$, there exists some K and a finite extension μ of ν such that $\mu \in \mathcal{J}_D^K$.* □

Next, a result stronger than Lemma 4.4 is established in that it is no longer an a prior requirement that $\mathcal{J}_D^s$ be nonempty for some s.

LEMMA 4.5. *Let $D \subseteq \mathcal{M}_1$ be compact. Then for any $j \ge 1$ and $\nu^j = (\nu_0, \ldots, \nu_{j-1}) \in \mathbb{R}^{mj}$, there exists a finite extension μ^k of ν^j such that $\mu^k \in \mathcal{J}_D^k$.* □

Sketch of the proof of Proposition 4.2. Let D be a compact subset of $\mathcal{M}_1$. To show the density property of $\mathcal{J}_D$, let $\nu = (\nu_0, \nu_1, \ldots) \in \mathbb{R}^{m,\infty}$, and let W be a neighborhood of ν. Without loss of generality, assume that

$$W = W_0 \times W_1 \times \cdots \times W_{l-1} \times \mathbb{R}^m \times \mathbb{R}^m \times \cdots,$$

where for each $0 \le i \le l-1$, W_i is an open subset of $\mathbb{R}^m$. By Lemma 4.5, there exists a finite extension μ^N of $\nu^l := (\nu_0, \ldots, \nu_{l-1})$ such that $\mu^N \in \mathcal{J}_D^N$. Note that for every extension $\sigma \in \mathbb{R}^{m,\infty}$ satisfying $\sigma_i = \mu_i$ for $0 \le i \le N-1$, it holds that $\sigma \in \mathcal{J}_D$ and $\sigma \in W$. This shows that $W \cap \mathcal{J}_D \neq \emptyset$.

To show that $\mathcal{J}_D$ is open, let $\nu = (\nu_0, \nu_1, \ldots) \in \mathcal{J}_D$. Then for each $p \in D$, there exists some i_p such that $\left[c_p \circ c_u(\nu^{i_p})\right]_{i_p} \neq 0$. By continuity, there exist a neighborhood $\mathcal{N}_p$ such that $\left[c_q \circ c_u(\nu^{i_p})\right]_{i_p} \neq 0$ for each $q \in \mathcal{N}_p$. Applying the compactness property of D, one sees that there exists an N such that $\left[c_q \circ c_u(\nu^N)\right]_N \neq 0$ for all $q \in D$. Again, by continuity, for each $q \in D$, there exist a neighborhood $B_q \subseteq \mathcal{M}_1$ of q and a neighborhood $U_{q,\nu^N} \subseteq \mathbb{R}^{m \times N}$ such that $\left[c_{q'} \circ c_u(\sigma^N)\right]_N \neq 0$ for all $q' \in B_q$ and all $\sigma^N \in U_{q,\nu^N}$. Using the compactness property of D once more, one sees that $D \subseteq \bigcup_{i=1}^L B_{q_i}$ for some $q_1, q_2, \ldots q_L$. Let $U_{\nu^N} = \bigcap_{i=1}^L U_{q_i,\nu^N}$. Since each U_{q_i,ν^N} is open, U_{ν^N} is an open subset of $\mathbb{R}^{m \times N}$. Moreover, $U_{\nu^N} \subseteq \mathcal{J}_D^N$. Finally, the proof is completed by letting $\mathcal{U} = U_{\nu^N} \times \mathbb{R}^{m,\infty}$. Note that $\mathcal{U}$ is an open set containing ν since

U_{ν^N} is an open set containing ν^N. For each $\sigma \in \mathcal{U}$, the restriction σ^N of σ is in U_{ν^N}, and hence $\sigma^N \in \mathcal{J}_D^N$. This implies that $\sigma \in \mathcal{J}_D$. Therefore, $\mathcal{U} \subseteq \mathcal{J}_D$, and ν is an interior point of $\mathcal{J}_D$.

5. Universal Smooth Inputs. For each $T > 0$, consider $C^\infty[0,T]$, the set of smooth functions from $[0,T]$ to $\mathbb{R}^m$, endowed with the Whitney topology. Note that this is a finer topology than the subspace topology induced from $C^0[0,T]$. Since $C^\infty[0,T]$ is a Baire space (see e.g. [3]), a generic subset of $C^\infty[0,T]$ is dense.

Given system (2.3), define for $T > 0$

$$\Omega_o^T = \{u \in C^\infty[0,T] : \ p \not\sim_u p_o \quad \forall\, p \in \mathcal{M}_1\}.$$

If $\mu \in \mathcal{J}$, then the corresponding u is an element of Ω_o^T. Hence, as a consequence of Theorem 4.1, $\Omega_o^T \neq \emptyset$. However, one can prove the following stronger result.

THEOREM 5.1. *Consider system (2.3) under Assumption A_0. Then for each $T > 0$, Ω_o^T is a generic subset of $C^\infty[0,T]$.*

To prove Theorem 5.1, define for any $S \subseteq \mathcal{M}_1$,

$$\Omega_o^T(S) = \{u \in C^\infty[0,T] : \ p \not\sim_u p_o \quad \forall\, p \in S\}.$$

LEMMA 5.2. *For each compact subset D of $\mathcal{M}_1$ and each $T > 0$, $\Omega_o^T(D)$ is open and dense in $C^\infty[0,T]$.*

Proof. Consider a compact subset D of $\mathcal{M}_1$, and let $T > 0$ be given. Suppose $u \in C^\infty[0,T]$, and let $\mathcal{W}$ be a neighborhood of u. Without loss of generality, one may assume that

$$\mathcal{W} = \Big\{ v \in C^\infty[0,T] : \ \|v^{(i)} - u^{(i)}\|_\infty < r, \ 0 \le i \le J-1 \Big\}$$

for some $J \ge 1$ and $r > 0$. Let $\mu \in \mathbb{R}^{m,\infty}$ be such that $\mu_i = u^{(i)}(0)$. By Lemma 4.5, there exists some $K \ge J$ and a finite extension ν^K of μ^J such that $\nu^K \in \mathcal{J}_D^K$. It can be shown that there exists some $w_0 \in C^\infty[0,T]$ such that:

(i) $w_0^{(i)}(0) = 0$ for all $i \le J-1$;
(ii) $w_0^{(i)}(0) = \nu_i - \mu_i$ for all $J \le i \le K-1$;
(iii) $\|w_0^{(i)}\|_\infty < \delta$ for $0 \le i \le J-1$.

Let $w(t) = u(t) + w_0(t)$. Then $w \in \mathcal{W}$, and with $\sigma_i = w^{(i)}(0)$, it holds that $\sigma \in \mathcal{J}_D$. Consequently, $w \in \mathcal{W}$. This shows that $\Omega_o^T(D)$ is dense.

To show that $\Omega_o^T(D)$ is open, let $u \in \Omega_o^T(D)$. Then for each $p \in D$, $y(t_p, p, u) \neq 0$ for some $t_p \in [0,T]$. By continuity, there exist a neighborhood $\mathcal{N}_p$ of p and a neighborhood U_p of u in $C^0[0,T]$ such that $y(t_p, q, v) \neq 0$ for all $q \in \mathcal{N}_p$ and $v \in U_p$. Again, by the compactness of D, one sees that there exist some $p_1, \ldots, p_N$ such that $D = \bigcup_{i=1}^{N} \mathcal{N}_{p_i}$. Let $\mathcal{O} = \bigcap_{i=1}^N U_{p_i}$. Then, for each $q \in D$ and each $v \in \mathcal{O}$, $y(t_{p_i}, q, v) \neq 0$ for some i. Hence, $\mathcal{O} \subseteq \Omega_o^T(D)$. Since $\mathcal{O}$ is open in $C^0[0,T]$, $\mathcal{O}$ is open in $C^\infty[0,T]$. This shows that $\Omega_o^T(D)$ is open in $C^\infty[0,T]$. ∎

Theorem 5.1 facilitates the characterization of universal inputs for distinguishability. Consider system (2.3) under the following assumption.

Assumption A_1. For each pair of points $(p,q) \in \mathcal{M} \times \mathcal{M}$ and any polynomial input u such that

$$\frac{d^k}{dt^k}\bigg|_{t=0^+} h(\varphi(t,p,u)) = \frac{d^k}{dt^k}\bigg|_{t=0^+} h(\varphi(t,q,u)), \quad k \ge 0,$$

it follows that $p \sim_u q$.

For system (2.3), define for $T > 0$,

$$\Omega^T = \{u \in C^\infty[0,T] : \ p \not\sim_u q \ \ \forall\,(p,q) \text{ such that } p \not\sim q\}.$$

The following corollary is a consequence of Theorem 5.1.

COROLLARY 5.3. *Consider system (2.3) under Assumption A_1. Then for each $T > 0$, Ω^T is a generic subset of $C^\infty[0,T]$.*

6. Conclusions. A proof for the existence of smooth universal inputs for a given smooth system was outlined in terms of formal power series and Fliess operators. The methodology is more direct in many ways than the existing proof and gives additional insight into this fundamental result.

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