CAUSAL INVERTIBILITY ON HILBERT RESOLUTION SPACES*

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Abstract. This note presents a necessary and sufficient condition for the inverse of a bounded causal operator to be causal. The abstract notion of causality employed is defined with respect to a so-called resolution of the idenity on the ambient Hilbert space; i.e. the operators considered map (linearly) between Hilbert resolution spaces. The causal invertibility condition established here involves a *family* of Toeplitz operators and the Fredholm index. For convolutions on $\boldsymbol{L}_2(-\infty,\infty)$, it is verified that the condition specializes to a frequency-domain winding number constraint. Indeed, the main result is relevant within the context of extending the ν-gap metric to time-varying systems.

Key words. Causal invertibility, linear systems

AMS subject classifications. 93C05, 93C25, 47L30

1. Introduction.

In various abstract settings, feedback stability is equivalent to the causal invertibility of a bounded causal operator [8, 10, 5]. This note establishes a *necessary* and *sufficient* condition for the inverse of a bounded causal operator to be causal, in terms of a generic resolution of the identity for imparting a time structure to the ambient Hilbert space; i.e. the resolution may be discrete or continuous and finite or forward/backward semi-infinite, even doubly-infinite. Ulitmately, the causal invertibility condition involves a corresponding *family* of Toeplitz operators and the Fredholm index. For causal convolution on the doubly-infinite $\boldsymbol{L}_2(-\infty,\infty)$, paired with the usual truncation based resolution of the identity, this condition specializes to a familiar winding number constraint on the transfer function associated with the impulse response. This confirms suitability of the condition for analysis aimed at extending the ν-gap metric [11, 4] to time-varying systems; see [5].

The related topological concept of *degree* is introduced in [8, Chapter 5]. While a degree condition is known to be sufficient for causal invertibility, in general it is not known to be necessary. In [1], a very different approach involving modification of the underlying resolution of the identity is used to characterize causal invertibility. No such modification is employed in this paper. In fact, the Toeplitz operators that arise here are all defined with respect to the original resolution of the identity.

The paper is structured as follows. Section 2 recalls aspects of operators on Hilbert resolution spaces. In Section 3, the causal invertibility condition is established. Consistency with a familiar winding number condition for causal convolutions is shown in Section 4. This is followed by some concluding remarks in Section 5.

2. Operators on Hilbert resolution spaces.

This section defines the notation and concepts required to formalize the main causal invertibility result. Much is taken from [8]. Suppose $\mathbb{T}$ is a linearly ordered set with a maximal element $t^{\max}$ and minimal element $t^{\min}$.

Definition 2.1. *A family $\mathscr{Q} = \{Q_\tau : \tau \in \mathbb{T}\}$ of orthogonal projections on a Hilbert space $\mathcal{H}$ is called a* resolution of the identity *if*

1. *$Q_t(\mathcal{H}) \supset Q_\tau(\mathcal{H})$ whenever $t \geq \tau$ for all $\tau \in \mathbb{T}$;*
2. *$Q_{t^{\min}} = 0$, $Q_{t^{\max}} = I$; and*
3. *$\mathscr{Q}$ is closed in that given a sequence $\{Q_{\tau_n}\} \subset \mathscr{Q}$, $Q_{\tau_n} \to Q$ implies $Q \in \mathscr{Q}$.*

*This work was supported in part by the Australian Research Council (DP110103778).

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Given a resolution of the identity $\mathscr{Q}$, on a Hilbert space $\mathcal{H}$, the pair $(\mathcal{H}, \mathscr{Q})$ is called a *Hilbert resolution space*. The family of projections $\mathscr{Q}$ imparts a time structure on $\mathcal{H}$. Given $\tau \in \mathbb{T}$, the subspace $\mathcal{H}^\tau := Q_\tau(\mathcal{H})$ is to be interpreted as the part before time τ and the subspace $\mathcal{H}_\tau := P_\tau(\mathcal{H})$, where $P_\tau := I - Q_\tau$, is to be interpreted as the part after time τ. Note that $\mathcal{H}_\tau$ is the orthogonal complement of $\mathcal{H}^\tau$ and that $\mathcal{H}_\tau \supset \mathcal{H}_t$, $\mathcal{H}^\tau \subset \mathcal{H}^t$ and $P_\tau P_t = P_t$ (which follows from the first property of $\mathscr{Q}$) for all $t \geq \tau$. Note that $\mathbb{T}$ may be continuous or discrete. Moreover, it may be finite, semi-infinite or doubly-infinite. For example: $\mathbb{T} = \{n \in \mathbb{Z} \,|\, n \geq 0\} \cup \{+\infty\}$ is a forward semi-infinite discrete time set with $t^{\min} = 0$ and $t^{\max} = +\infty$; $\mathbb{T} = \{-\infty\} \cup (-\infty, T)$ is a backward infinite time set with $t^{\min} = -\infty$ and $t^{\max} = T \in \mathbb{R} \cup +\infty$; and $\mathbb{T} = \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ is a doubly-infinite continuous time set with $t^{\min} = -\infty$ and $t^{\max} = +\infty$ (this is both forward and backward infinite).

Let $\mathscr{B}$ denote the Banach algebra of all bounded linear operators on $\mathcal{H}$ and $\mathscr{B}^{-1}$ the subset of such that are one-to-one and onto $\mathcal{H}$, whereby a bounded inverse exists by the inverse-mapping theorem [2, Theorem 5.6]. Given $M \in \mathscr{B}$ and $\tau \in \mathbb{T}$, define the Toeplitz operators

$$T_\tau(M) := P_\tau M|_{\mathcal{H}_\tau} \quad \text{and} \quad U_\tau(M) := Q_\tau M|_{\mathcal{H}^\tau} \tag{2.1}$$

and the Hankel operators

$$H_\tau(M) := P_\tau M|_{\mathcal{H}^\tau} \quad \text{and} \quad J_\tau(M) := Q_\tau M|_{\mathcal{H}_\tau}, \tag{2.2}$$

where $M|_{\mathcal{V}}$ denotes the restriction of M to $\mathcal{V} \subset \mathcal{H}$. Observe that

$$M = \begin{bmatrix} P_\tau & Q_\tau \end{bmatrix} \begin{bmatrix} T_\tau(M) & H_\tau(M) \\ J_\tau(M) & U_\tau(M) \end{bmatrix} \begin{bmatrix} P_\tau \\ Q_\tau \end{bmatrix} \quad \forall\, \tau \in \mathbb{T}.$$

Moreover, for $M, N \in \mathscr{B}$ it follows that

$$T_\tau(NM) = T_\tau(N) T_\tau(M) + H_\tau(N) J_\tau(M) \quad \forall\, \tau \in \mathbb{T}. \tag{2.3}$$

Let $\mathscr{C}$ denote the set of $M \in \mathscr{B}$ such that $M(\mathcal{H}_\tau) \subset \mathcal{H}_\tau$ for all $\tau \in \mathbb{T}$; i.e. the set of *causal* operators. This is a weakly closed Banach algebra with identity [8, Property 6] and $\mathscr{C} \subset \mathscr{B}$. Also, let $\mathscr{C}^{-1}$ denote the set of operators $M \in \mathscr{B}^{-1}$ for which the inverse $M^{-1} \in \mathscr{C}$.

LEMMA 2.2. *If $M \in \mathscr{C}$, then for any $\tau \in \mathbb{T}$ the Toeplitz operator $T_\tau(M) = M|_{\mathcal{H}_\tau}$ and the Hankel operator $J_\tau(M) = 0$, whereby $T_\tau(NM) = T_\tau(N) T_\tau(M)$ for $N \in \mathscr{B}$.*

Proof. The first two claims follow directly from the definitions and the last holds by the second in view of (2.3). □

In the subsequent analysis, given a $\tau \in \mathbb{T}$, the Toeplitz operator $T_\tau(M)$ associated with an $M \in \mathscr{B}$ is said to be causal if $\mathcal{H}_t$ is invariant under $T_\tau(M)$ for all $t \geq \tau$. If $M \in \mathscr{C}$, then $T_\tau(M)$ is causal for all $\tau \in \mathbb{T}$. The converse also holds, as shown below.

LEMMA 2.3. *Given $M \in \mathscr{B}$, if $T_\tau(M)$ is causal for all $\tau \in \mathbb{T}$, then $M \in \mathscr{C}$.*

Proof. Suppose to the contra-positive that $M \notin \mathscr{C}$. Then there exist $t \in \mathbb{T}$ and $u_t \in \mathcal{H}_t$, such that $x := Q_t y_t$ is non-zero, where $y_t := M u_t$; note $t > t^{\min}$ because $Q_{t^{\min}} = 0$. Moreover, $z := P_\tau x \neq 0$ for some $\tau \in \mathbb{T}$ with $\tau < t$ (take e.g. $\tau = t^{\min}$). Let $y_{t,\tau} := P_\tau y_t$, whereby $Q_t y_{t,\tau} = (I - P_t) P_\tau y_t = (I - P_\tau P_t) P_\tau y_t = P_\tau (I - P_t) y_t = P_\tau x = z \neq 0$. Also note that $y_{t,\tau} = P_\tau M P_\tau u_t = T_\tau(M) u_t$, as $P_\tau u_t = u_t$ for $\tau < t$. Since $Q_t u_t = 0$, this contradicts the causality of $T_\tau(M)$. □

Remark 1. *Note that Lemma 2.3 actually holds trivially since $T_{t^{\min}}(M) = M$. The proof presented reveals the possibility of relaxing the sufficient condition for specific instances of the index set $\mathbb{T}$. For example, when the index set is backward semi-infinite (continuous or discrete), the sufficient condition in Lemma 2.3 can be stated as $T_\tau(M)$ causal for all $\tau \in \mathbb{T} \setminus \{t^{\min}\}$. For this case, τ in the proof can be chosen such that $t^{\min} < \tau < t$, since otherwise $P_\tau x = 0$ for all $\tau > t^{\min}$ would imply $x = 0$. Note that when $\mathbb{T}$ is discrete and forward semi-infinite, it may be necessary to choose $\tau = t^{\min}$ in the proof. However, in this case it is well-known that invertibility is equivalent to causal invertibility [7].*

Before proceeding to characterize when $M \in \mathscr{B}^{-1} \cap \mathscr{C}$ is such that $M^{-1} \in \mathscr{C}$, it is instructive to recall the following definition.

Definition 2.4. *An operator $M \in \mathscr{B}$ is said to be Fredholm if the dimensions of* $\ker(M) := \{u \in \mathcal{H} : Mu = 0\}$ *and* $\operatorname{coker}(M) := \mathcal{H}/M(\mathcal{H}) = \{[u] : u \in \mathcal{H}\}$ *are both* finite, *where $[u] := u + M(\mathcal{H})$ denotes the equivalence class of u defined by the equivalence relation $u_1 \sim u_2$ if $u_1 - u_2 \in M(\mathcal{H})$.*

If $M \in \mathscr{B}$ is Fredholm, $M(\mathcal{H})$ is necessarily closed since $\operatorname{coker}(M)$ is finite dimensional [9, Corollary XI.2.3] and so $\dim \operatorname{coker}(M) = \dim M(\mathcal{H})^\perp = \dim \ker(M^*)$, where dim denotes the dimension of a subspace, $\perp$ denotes orthogonal complement and $M^* \in \mathscr{B}$ denotes the Hilbert adjoint of M. In this case, the Fredholm index is defined by $\operatorname{ind}(M) := \dim \ker(M) - \dim \operatorname{coker}(M) = \dim \ker(M) - \dim \ker(M^*)$. Note that if M is one-to-one and onto $\mathcal{H}$, then it is Fredholm with $\operatorname{ind}(M) = 0$.

3. Causal invertibility. This section presents the main causal invertibility results of the paper.

Theorem 3.1. *Given $M \in \mathscr{B}^{-1} \cap \mathscr{C}$, the following are equivalent:*

(*i*) *$M \in \mathscr{C}^{-1}$;*

(*ii*) *$T_\tau(M)$ is onto H_τ for all $\tau \in \mathbb{T}$; and*

(*iii*) *$T_\tau(M)$ is Fredholm with* $\operatorname{ind}(T_\tau(M)) = 0$ *for all $\tau \in \mathbb{T}$.*

Proof. First note the following consequences of the hypothesis $M \in \mathscr{B}^{-1} \cap \mathscr{C}$. In particular, since $M^{-1} \in \mathscr{B}$ it follows that $\mu(M) := \inf_{\|u\|=1} \|Mu\| > 0$, whereby $\mu(T_\tau(M)) > 0 \,\forall\, \tau \in \mathbb{T}$ since $M \in \mathscr{C}$, which implies $T_\tau(M) = M|_{\mathcal{H}_\tau} \,\forall\, \tau \in \mathbb{T}$ in view of Lemma 2.2. That is, $T_\tau(M)$ is one-to-one for all $\tau \in \mathbb{T}$.

(*i*) $\Rightarrow$ (*ii*): Since $M^{-1} \in \mathscr{C}$, it follows by Lemma 2.2 that $I|_{\mathcal{H}_\tau} = T_\tau(MM^{-1}) = T_\tau(M)T_\tau(M^{-1})$, whereby $T_\tau(M)$ is onto $\mathcal{H}_\tau$ for all $\tau \in \mathbb{T}$, as claimed.

(*i*) $\Leftarrow$ (*ii*): As noted above, $T_\tau(M)$ is also one-to-one and thus, $(T_\tau(M))^{-1} : \mathcal{H}_\tau \to \mathcal{H}_\tau$ is bounded by the inverse-mapping theorem [2, Theorem 5.6]. Moreover, $(T_\tau(M))^{-1} = T_\tau(M^{-1})$ because $M \in \mathscr{C}$, which in view of Lemma 2.2 implies $I|_{\mathcal{H}_\tau} = T_\tau(M^{-1}M) = T_\tau(M^{-1})T_\tau(M)$. Therefore, $T_\tau(M^{-1})$ is causal for all $\tau \in \mathbb{T}$ in the sense that $\mathcal{H}_t$ is invariant under $T_\tau(M^{-1})$ for all $t > \tau$; by Lemma 2.3 this implies $M^{-1} \in \mathscr{C}$ as claimed. To see that $T_\tau(M^{-1})$ is causal for any given $\tau \in \mathbb{T}$, first note that $T_t(M) = T_\tau(M)|_{\mathcal{H}_t} \,\forall\, t > \tau$; specifically, using $M \in \mathscr{C}$, Lemma 2.2 and the fact that $P_\tau P_t = P_t$ for all $\tau < t$, it follows that

$$T_t(M) = P_t M|_{\mathcal{H}_t} = P_\tau M P_t|_{\mathcal{H}_t} = P_\tau M P_\tau P_t|_{\mathcal{H}_t} = T_\tau(M)|_{\mathcal{H}_t}. \tag{3.1}$$

Now suppose to the contra-positive that there exist $y \in \mathcal{H}_\tau$ and $t > \tau$ for which $Q_t y = 0$ (i.e. $y \in \mathcal{H}_t$) and $Q_t(T_\tau(M))^{-1}y = Q_t T_\tau(M^{-1})y \neq 0$; in other words, suppose $T_\tau(M^{-1})$ is not causal. Noting that $y \in \mathcal{H}_t \subset \mathcal{H}_\tau$, let $u_\tau := (T_\tau(M))^{-1}y = T_\tau(M^{-1})y$ and $u_t := (T_t(M))^{-1}y = T_t(M^{-1})y$. Then using (3.1) gives $T_\tau u_\tau = y = T_t u_t = T_\tau u_t$. As such, $T_\tau(u_\tau - u_t) = 0$, which implies $u_\tau = u_t \in \mathcal{H}_t$ because $\ker(T_\tau(M)) = \{0\}$. This contradicts $Q_t u_\tau \neq 0$.

(ii) ⇔ *(iii)*: This equivalence follows by the definition of a Fredholm operator since, by hypothesis, $T_\tau(M)$ is one-to-one for all $\tau \in \mathbb{T}$. Specifically, if $T_\tau(M)$ is also onto $\mathcal{H}_\tau$, then it is boundedly invertible by the inverse-mapping theorem, which as noted above implies $T_\tau(M)$ is Fredholm with $\mathrm{ind}(T_\tau(M)) = 0$, giving *(iii)*. Conversely, $\mathrm{ind}(T_\tau(M)) = 0$ implies $\dim \mathrm{coker}(T_\tau(M)) = 0$, as $\ker(T_\tau(M)) = \{0\}$ by hypothesis, so that $T_\tau(M)$ is onto $\mathcal{H}_\tau$, yielding *(ii)*. ◻

Remark 2. *The question of causal invertibility is studied in [8, Chapter 5]. There, a sufficient condition for (i) in Theorem 3.1 is given in terms of the so-called* degree $\deg[M]$ *of* $M \in \mathscr{B}^{-1} \cap \mathscr{C}$, *which is defined to be the equivalence class of* M *associated with the following equivalence relation:* $M_1 \sim M_2$ *if there exists a uniformly continuous function* $f : [0,1] \to \mathscr{B}^{-1} \cap \mathscr{C}$ *such that* $f(0) = M_1$ *and* $f(1) = M_2$. *Specifically, the sufficient condition is* $\deg[M] = 0 := \deg[I]$. *It is noted in [8, Remark 11] that* $\deg[M] = 0$ *is not known to be a necessary condition in general. In view of the necessity of Theorem 3.1, the condition involving a family of Toeplitz operators and a Fredholm index condition in (iii) is clearly implied by* $\deg[M] = 0$. *To date, a proof of the converse has eluded the authors.*

4. The case of convolution operators on $\boldsymbol{L}_2(-\infty,\infty)$. Let $\hat{\mathscr{A}}(\sigma)$ denote the algebra of transfer functions obtained by taking the Laplace transform of functions in the form of $\kappa(t) = \kappa_r(t) + \kappa_a\delta(t)$, where $e^{-\sigma t}\kappa_r(t) \in \boldsymbol{L}_1[0,\infty) := \{f : \mathbb{R} \to \mathbb{R} : \int_0^\infty |f(t)|\,dt < \infty;\ f(t) = 0\,\forall t \in (-\infty,0)\}$, $\kappa_a \in \mathbb{R}$, and $\delta(\cdot)$ is the Dirac distribution; see [3]. For convenience let $\hat{\mathscr{A}} := \hat{\mathscr{A}}(0)$. A transfer function $\hat{M} \in \hat{\mathscr{A}}$ is continuous and bounded on the imaginary axis, holomorphic in $\mathbb{C}_{0+} := \{s \in \mathbb{C} : \Re e(s) > 0\}$ and constant for $|s| = \infty$ [6, Theorem A.7.47]. One may associate with $\hat{M} \in \hat{\mathscr{A}}$ a *causal* convolution $M : \boldsymbol{L}_2(\mathbb{R}) \to \boldsymbol{L}_2(\mathbb{R})$ defined by

$$(Mu)(t) = \int_{-\infty}^{t} \kappa_r(t-s)u(s)\,ds + \kappa_a u(t) \quad \forall t \in \mathbb{R},$$

where $\boldsymbol{L}_2(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : \int_{-\infty}^{\infty} |f(t)|^2 dt < \infty\}$ and this is bounded, with induced norm $\|M\| = \|M\|_\infty := \sup_{\omega\in\mathbb{R}} \sigma_{\max}(\hat{M}(j\omega))$, where $j := \sqrt{-1}$. Indeed, let $\boldsymbol{L}_2(j\mathbb{R})$ denote the frequency-domain space that is isometrically isomorphic to $\boldsymbol{L}_2(\mathbb{R})$ via the Fourier transform [6, Theorem A.6.13] and observe that M is equivalent to the multiplication operator $\hat{u} \in \boldsymbol{L}_2(j\mathbb{R}) \mapsto \hat{y} \in \boldsymbol{L}_2(j\mathbb{R})$ defined by $\hat{y}(j\omega) := \hat{M}(j\omega)\hat{u}(j\omega)\,\forall\,\omega \in \mathbb{R}$, which has induced norm $\|\hat{M}\|_\infty$ [6, Lemmas A.6.5 and A.6.26].

For any interval $\mathbb{I} \subset \mathbb{R}$, the subspace of all $u \in \boldsymbol{L}_2(\mathbb{R})$ such that $u(t) = 0$ for $t \in \mathbb{R}\setminus\mathbb{I}$ is denoted by $\boldsymbol{L}_2(\mathbb{I})$. With $\mathbb{R}_\tau := (\tau,\infty)$ and $\mathbb{R}^\tau := (-\infty,\tau)$, note that $\boldsymbol{L}_2(\mathbb{R}) = \boldsymbol{L}_2(\mathbb{R}_\tau) \oplus \boldsymbol{L}_2(\mathbb{R}^\tau)\ \forall\ \tau \in \mathbb{R}$, where $\oplus$ denotes orthogonal sum. Let the orthogonal projection from $\boldsymbol{L}_2(\mathbb{R})$ onto $\boldsymbol{L}_2(\mathbb{R}_\tau)$ be denoted by P_τ and let $Q_\tau := I - P_\tau$. Then in accordance Definition 2.1, $\mathscr{Q} := \{Q_\tau : \tau \in \{-\infty\} \cup \mathbb{R} \cup \{\infty\}\}$, where $Q_{-\infty} := 0$ and $Q_\infty := I$, is a resolution of the identity and the pair $(\boldsymbol{L}_2(\mathbb{R}), \mathscr{Q})$ is a Hilbert resolution space. The causal convolution M associated with $\hat{M} \in \hat{\mathscr{A}}$ is clearly an element of the corresponding $\mathscr{C}$. It follows that $M \in \mathscr{B}^{-1}$ if, and only if, $\hat{M}(j\omega) \neq 0$ for all $\omega \in \mathbb{R}\cup\{\infty\}$. In this case, Theorem 3.1 can be used to characterize when $M \in \mathscr{C}^{-1}$, as described below.

According to [9, Theorem XII.3.1], the Toeplitz operator $T_0(M) : \boldsymbol{L}_2(\mathbb{R}_0) \to \boldsymbol{L}_2(\mathbb{R}_0)$, which satisfies

$$(T_0(M)u)(t) = \int_0^t \kappa_r(t-s)u(s)\,ds + \kappa_a u(t) \quad \forall\, t \in \mathbb{R}_0,$$

is Fredholm if, and only, if $\hat{M}(j\omega) \neq 0$ for all $\omega \in \mathbb{R} \cup \{\infty\}$. In this case, $\mathrm{ind}(T_0(M)) = -\mathrm{wno}(\hat{M})$, where $\mathrm{wno}(\hat{M})$ is the net increase in $(1/2\pi)\arg(M(j\omega))$ as ω decreases from $+\infty$ to $-\infty$; i.e. a winding number. Given $\tau \in \mathbb{R}$ and $u \in \boldsymbol{L}_2(\mathbb{R}_\tau)$, let $u_\tau \in \boldsymbol{L}_2(\mathbb{R}_0)$ be defined as $u_\tau(s) := u(s+\tau)$ for all $s \in \mathbb{R}$ and note that

$$\begin{aligned}(T_\tau(M)u)(t+\tau) &= \int_\tau^{t+\tau} \kappa_r(t+\tau-s)u(s)\,ds + \kappa_a u(t+\tau) \\ &= \int_0^t \kappa_r(t-v)u_\tau(v)\,dv + \kappa_a u_\tau(t) = (T_0(M)u_\tau)(t)\end{aligned}$$

for all $t > 0$. As such, $T_\tau(M)$ is Fredholm if, and only if, $T_0(M)$ is Fredholm and in this case, $\mathrm{ind}(T_\tau(M)) = \mathrm{ind}(T_0(M))$. In view of this, the following hold. By Theorem 3.1, when $\hat{M}(j\omega) \neq 0\,\forall\,\omega \in \mathbb{R} \cup \{\infty\}$ it holds that $M^{-1} \in \mathscr{C}$ if, and only if, $\mathrm{wno}(\hat{M}) = 0$. In fact, it is shown that $\hat{M}(j\omega) \neq 0\,\forall\,\omega \in \mathbb{R} \cup \{\infty\}$ and $\mathrm{wno}(\hat{M}) = 0$ if, and only if, $\hat{M}^{-1} \in \hat{\mathscr{A}}$ [4, Lemma 1]. Theorem 7.1.5 in [6] contains related results expressed in terms of conditions over the whole of the right-half plane.

5. Conclusion. A necessary and sufficient condition is established for the inverse of a bounded causal operator to be causal. The condition involves a family of Toeplitz operators and the Fredholm index. This specializes to a frequency-domain winding number condition in the case of convolution operators on $\boldsymbol{L}_2(-\infty, \infty)$. This plays an important role in the generalization of the ν-gap metric [11, 4] to linear time-varying systems in [5].

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