ALGEBRAIC CHARACTERIZATION OF FREE DIRECTIONS OF SCALAR n-D AUTONOMOUS SYSTEMS*

DEBASATTAM PAL† AND HARISH K. PILLAI‡

Abstract. In this paper, restriction of scalar n-D systems to 1-D subspaces has been considered. It has been shown that for general n-D systems there can be *free* subspaces, meaning every 1-D trajectory can be obtained by restricting trajectories in the original system. This paper gives an algebraic characterization for all free directions in terms of intersection ideals.

Key words. n-D systems, free directions, restriction to 1-D subspace, intersection ideal.

AMS subject classifications.

1. Introduction and preliminaries. In this paper we consider systems of linear partial differential equations (PDEs) with constant real coefficients over n independent variables, denoted here by $\mathbf{x} := (x_1, x_2, \ldots, x_n)$. Throughout the paper, we consider only one dependent variable, denoted here by w. We use the notation ∂_i for the ith partial derivative $\frac{\partial}{\partial x_i}$, and the symbol ∂ to denote the n-tuple $\{\partial_1, \partial_2, \ldots, \partial_n\}$. Following Willems [8], we call the solution sets of such systems of PDEs as *behaviors* and denote them by $\mathfrak{B}$. Thus

$$\mathfrak{B} := \{w \in \mathcal{W} \mid f_1(\partial)w = f_2(\partial)w = \cdots = f_r(\partial)w = 0\}, \tag{1.1}$$

where $f_i(\partial)$, for $1 \leqslant i \leqslant r$, are polynomial differential operators with constant real coefficients. We denote by $\mathbb{R}[\partial]$ the polynomial ring in $\{\partial_1, \partial_2, \ldots, \partial_n\}$ with real coefficients. Thus $f_i(\partial) \in \mathbb{R}[\partial]$ for $1 \leqslant i \leqslant r$. (We shall refer to these behaviors as *scalar autonomous behaviors* in the sequel.) The symbol $\mathcal{W}$ in equation (1.1) denotes the space of trajectories where solutions to the given system of equations are sought. In this paper we consider only real entire analytic solutions of exponential type, which is defined as follows.

Definition 1. *We denote by $\mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$ the set of all formal power series in n variables*

$$w(\mathbf{x}) = \sum_{\nu \in \mathbb{N}^n} \frac{w_\nu}{\nu!} \mathbf{x}^\nu,$$

where $\nu = (\nu_1, \nu_2, ..., \nu_n) \in \mathbb{N}^n$ is a multi-index, $\mathbf{x}^\nu$ means the monomial $x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n}$ and $\nu!$ denotes $\nu_1!\nu_2! \cdots \nu_n!$, with the sequence of real numbers $\{w_\nu\}_{\nu \in \mathbb{N}^n}$ being such that w is convergent everywhere, that is, $w(\mathbf{a}) \in \mathbb{R}$ for all $\mathbf{a} \in \mathbb{R}^n$.

Remark 2. In [3, 5], it has been shown that when the solution space $\mathcal{W} = \mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$, then two sets of equations, $f_1(\partial)w = f_2(\partial)w = \cdots = f_r(\partial)w = 0$ and $g_1(\partial)w = g_2(\partial)w = \cdots = g_s(\partial)w = 0$, give rise to the same behavior if and only if the ideals generated by $\{f_1(\partial), f_2(\partial), \ldots, f_r(\partial)\}$ and $\{g_1(\partial), g_2(\partial), \ldots, g_s(\partial)\}$ are equal. Thus scalar autonomous behaviors turn out to be in one-to-one correspondence with ideals in $\mathbb{R}[\partial]$. This motivates the following description of behaviors: given an ideal

*This work was supported by Bharti Centre for Communication, IIT Bombay.

†Department of Electrical Engineering, Indian Institute of Technology Bombay, Mumbai, India (debpal@ee.iitb.ac.in).

‡Department of Electrical Engineering, Indian Institute of Technology Bombay, Mumbai, India (hp@ee.iitb.ac.in).

$\mathcal{I} \subseteq \mathbb{R}[\partial]$,

$$\mathfrak{B} := \{w \in \mathfrak{Exp}(\mathbb{R}^n, \mathbb{R}) \mid f(\partial)w = 0 \text{ for all } f(\partial) \in \mathcal{I}\}.$$

The ideal $\mathcal{I}$ is called the *equation ideal* of $\mathfrak{B}$.

In this paper, we analyze the restriction of a behavior to a given 1-D subspace in the domain. A unique characteristic of systems with $n \geqslant 2$ is that for some 1-D subspaces every possible 1-D trajectory may be obtained by restricting trajectories in the original behavior to this subspace. We call the spanning vectors of such 1-D subspaces as *free directions*. See [1] where a similar issue has been addressed in the context of discrete nD systems. In order to make this idea of free directions precise, we first define restriction of $\mathfrak{B}$ to a given 1-D subspace. Note that given $0 \neq v \in \mathbb{R}^n$ and $w \in \mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$, the map $\mathbb{R} \ni t \mapsto w(vt) \in \mathbb{R}$ defines a trajectory in $\mathfrak{Exp}(\mathbb{R}, \mathbb{R})$. We denote this 1-D trajectory by $w(vt)$.

Definition 3. *Given $0 \neq v \in \mathbb{R}^n$ and a behavior $\mathfrak{B}$, by $\mathfrak{B}|_v$ we denote the following set of 1-D trajectories:*

$$\mathfrak{B}|_v := \{w(vt) \in \mathfrak{Exp}(\mathbb{R}, \mathbb{R}) \mid w \in \mathfrak{B}\}.$$

A given $0 \neq v \in \mathbb{R}^n$ is said to be a free direction *of a behavior $\mathfrak{B}$ if*

$$\mathfrak{B}|_v = \mathfrak{Exp}(\mathbb{R}, \mathbb{R}).$$

Example 4. As an example of free directions consider the following scalar system of PDEs: $\mathfrak{B} = \{w \in \mathfrak{Exp}(\mathbb{R}^3, \mathbb{R}) \mid \partial_2^2 w = \partial_3^2 w = \partial_1\partial_3 w - \partial_2 w = 0\}$. Clearly, any exponential trajectory of the form $w(x_1, x_2, x_3) = p(x_1)e^{\alpha x_1}$ with $p(x_1) \in \mathbb{R}[x_1]$ is a solution to the above system of equations. Every 1-D exponential function is of the above form. So, indeed, x_1-axis is a free direction.

Our main result Theorem 9 provides algebraic conditions equivalent to a given direction being free. The rest of this section provides some preliminary results crucial for proving Theorem 9. Then in Section 2 we state and prove the main result Theorem 9.

Proposition 5 below has been proved in [6]; this result will be important for us in proving Theorem 9. We need the following background to state Proposition 5. Given an ideal $\mathcal{I} \subseteq \mathbb{R}[\partial]$, and $0 \neq v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$, we define the v-intersection ideal as $\mathcal{I}_v := \mathcal{I} \cap \mathbb{R}[\langle v, \partial\rangle]$, where $\mathbb{R}[\langle v, \partial\rangle]$ denotes the $\mathbb{R}$-algebra generated by the linear polynomial $v_1\partial_1 + v_2\partial_2 + \cdots + v_n\partial_n$. Clearly, $\mathcal{I}_v$ is an ideal in $\mathbb{R}[\langle v, \partial\rangle]$. Related to this is the following 1-D behavior:

$$\mathfrak{B}_v := \{\widetilde{w} \in \mathfrak{Exp}(\mathbb{R}, \mathbb{R}) \mid f(\frac{\mathrm{d}}{\mathrm{d}t})\widetilde{w} = 0 \text{ for all } f(\langle v, \partial\rangle) \in \mathcal{I}_v\}. \tag{1.2}$$

Proposition 5. *Let $\mathfrak{B}$ be a scalar autonomous behavior with equation ideal $\mathcal{I}$ and let $0 \neq v \in \mathbb{R}^n$ be given. Further, let $\mathfrak{B}|_v$ be as defined in Definition 3 and $\mathfrak{B}_v$ be as defined by equation (1.2) above. Then we have*

$$\mathfrak{B}|_v \subseteq \mathfrak{B}_v.$$

In this paper we make crucial use of a Gröbner basis method of obtaining exponential type solutions of PDEs. In [3, 4] Oberst elaborated this method extensively and showed how it can be utilized to construct power series solutions to the Cauchy problems in PDEs. Algorithm 6 is a short description of this Gröbner basis method for formal integration of PDEs for the single dependent variable case (see [4] for the general case).

Algorithm 6.

Level-1
Input: A set of PDEs $f_1(\partial)w = f_2(\partial)w = \cdots = f_r(\partial)w = 0$.
Computation:

- Fix a term ordering $\prec$ in $\mathbb{R}[\partial]$.
- Compute a Gröbner basis $\mathcal{G}$ of the ideal $\mathcal{I} := \langle f_1, f_2, \dots, f_r\rangle$.
- Construct the set of standard monomials $\Gamma := \{\nu \in \mathbb{N}^n \mid \partial^\nu \notin \mathrm{in}_\prec(\mathcal{I})\}$.

Output: *Standard monomial set* Γ.
Level-2
Input: *Initial data*: $\{w_\nu \in \mathbb{R}\}_{\nu\in\Gamma}$.
Computation:

for $\nu \notin \Gamma$

- Compute by division algorithm by $\mathcal{G}$ to obtain

$$\partial^\nu \equiv \sum_{i=1,\nu_i\in\Gamma}^{k<\infty} \alpha_i \partial^{\nu_i} \text{ modulo } \mathcal{I}.$$

- Set $w_\nu = \sum_{i=1}^k \alpha_i w_{\nu_i}$.

end
Output The sequence $w := \{w_\nu\}_{\nu\in\mathbb{N}^n}$.

In [3, 4] Oberst shows that the output of the above algorithm, when written in the power series form as $w = \sum_{\nu\in\mathbb{N}^n} \frac{w_\nu}{\nu!}\mathbf{x}^\nu$, is indeed a solution to the given set of PDEs, and conversely, every entire solution is obtained from this algorithm by giving different initial conditions $\{w_\nu\}_{\nu\in\Gamma}$, where Γ is the standard monomial set computed in level-1 of Algorithm 6. However, Algorithm 6 says nothing about convergence of the solution. In [4, 5], it was proved that if the initial data itself is an exponential trajectory then the solution obtained following Algorithm 6 is guaranteed to be an exponential one. We paraphrase this result in the following proposition; this will be crucial for us while proving the main result Theorem 9.

PROPOSITION 7. **(Theorems 24 and 26, [5])** *Given a set of PDEs $f_1(\partial)w = f_2(\partial)w = \cdots = f_r(\partial)w = 0$, and a term ordering $\prec$ of $\mathbb{R}[\partial]$, let Γ be the set of standard monomials. Further, let $w_{\mathrm{in}} := \{w_\nu\}_{\nu\in\Gamma}$ be an arbitrary sequence of real numbers indexed by Γ. With this w_{in} as the initial data, let $\{w_\nu\}_{\nu\in\mathbb{N}^n}$ be the output of Algorithm 6. Suppose the following formal power series*

$$\hat{w}(\mathbf{x}) := \sum_{\nu\in\Gamma} \frac{w_\nu}{\nu!}\mathbf{x}^\nu$$

obtained from w_{in} converges for all $\mathbf{x} \in \mathbb{R}^n$.

Then so does the power series

$$w(\mathbf{x}) := \sum_{\nu\in\mathbb{N}^n} \frac{w_\nu}{\nu!}\mathbf{x}^\nu$$

obtained from the solution of Algorithm 6. That is, $\hat{w}(\mathbf{x}) \in \mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$ implies $w(\mathbf{x}) \in \mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$. Keeping the above result in mind, we call an initial condition w_{in} (or $\hat{w}(\mathbf{x}) = \sum_{\nu\in\Gamma} \frac{w_\nu}{\nu!}\mathbf{x}^\nu$) *valid* if $\hat{w} \in \mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$.

REMARK 8. Let $\widetilde{w} = \sum_{\lambda\in\mathbb{N}} \frac{\widetilde{w}_\lambda}{\lambda!} t^\lambda \in \mathfrak{Exp}(\mathbb{R}, \mathbb{R})$ be any 1-D exponential trajectory. If we define $\Gamma_i := \{\nu \in \mathbb{N}^n \mid \nu = \lambda e_i, \lambda \in \mathbb{N}\}$, e_i being the standard ith basis vector in $\mathbb{R}^n$, and assume that for some term ordering we have $\Gamma_i \subseteq \Gamma$, then notice that the following initial condition is a valid one.

$$\hat{w}(\mathbf{x}) = \sum_{\nu\in\Gamma} \frac{w_\nu}{\nu!}\mathbf{x}^\nu, \text{ where } w_\nu = \begin{cases} \widetilde{w}_\lambda & \text{if } \nu \in \Gamma_i \text{ and } \nu = \lambda e_i \\ 0 & \text{otherwise} \end{cases}.$$

This is because if we denote by x_i the ith coordinate function, then $\mathfrak{Exp}(\mathbb{R},\mathbb{R}) \ni \widetilde{w}(t) \mapsto w(\mathbf{x}) := \widetilde{w}(x_i) \in \mathfrak{Exp}(\mathbb{R}^n,\mathbb{R})$ is an injection. Now, if indeed $\Gamma_i \subseteq \Gamma$, and w_{in} is chosen from a 1-D exponential trajectory $\widetilde{w}$, then Algorithm 6 guarantees that the corresponding solution, say w, when restricted to e_i, gives back $\widetilde{w}$. Since an initial condition can be freely chosen, it follows that for any 1-D exponential trajectory there exists a trajectory in the solution set of the PDEs whose restriction onto e_i is that 1-D trajectory. In other words, e_i is a free direction. We exploit this observation in the proof of Theorem 9.

2. Main result. Recall the definition of the v-intersection ideal of a given ideal in $\mathbb{R}[\partial]$. We will see in Theorem 9 below that $0 \neq v \in \mathbb{R}^n$ is a free direction if and only if the v-intersection ideal is the zero ideal.

THEOREM 9. *Let $\mathfrak{B}$ be a scalar autonomous behavior defined by the equation ideal $\mathcal{I} \subseteq \mathbb{R}[\partial]$ and let $0 \neq v \in \mathbb{R}^n$. Then the following conditions are equivalent:*

1. *v is a free direction of $\mathfrak{B}$.*
2. *The intersection ideal $\mathcal{I}_v := \mathcal{I} \cap \mathbb{R}[\langle v, \partial\rangle]$ is the zero ideal.*
3. *The $\mathbb{R}$-algebra homomorphism φ in the following commutative diagram is injective.*

$$\begin{tikzcd} \mathbb{R}[\partial] \arrow[r, two heads] & \mathbb{R}[\partial]/\mathcal{I} \\ \mathbb{R}[\langle v,\partial\rangle] \arrow[u] \arrow[ur, "\varphi"'] & \end{tikzcd}.$$

The next result is a technical lemma required in the proof of Theorem 9. The lemma deals with the effect on the equation ideal and the behavior due to a change of basis in the domain. This is closely related to the differential geometric notion of *push-forward* of a map between two differentiable manifolds to a map between the two tangent spaces. We give a short description of this notion below; details can be found in textbooks, see for example [2].

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear map. We call the coordinate functions of the domain and the codomain spaces $\mathbf{x}$ and $\mathbf{y}$, respectively. Then $\mathbf{x}$ and $\mathbf{y}$ are related by $\mathbf{y} = T\mathbf{x}$. This induces a map between the tangent spaces, $T^* : \mathcal{T}_{\mathbf{x}}\mathbb{R}^n \to \mathcal{T}_{\mathbf{y}}\mathbb{R}^n$, as follows. Let $\mathbf{y} \mapsto w(\mathbf{y})$ be in $\mathfrak{Exp}(\mathbb{R}^n,\mathbb{R})$. Define for all $1 \leqslant i \leqslant n$

$$\left(T^*\frac{\partial}{\partial x_i}\right)(w(\mathbf{y})) := \frac{\partial}{\partial x_i} w(T\mathbf{x}).$$

Let T be given by the matrix $T = [t_{ij}]_{1\leqslant i,j\leqslant n}$. Then it follows from the definition of T^* that $\left(T^*\frac{\partial}{\partial x_i}\right) y_j = \frac{\partial}{\partial x_i}\sum_{k=1}^n t_{jk}x_k = t_{ji}$. By varying j, we get $\left(T^*\frac{\partial}{\partial x_i}\right) = \sum_{j=1}^n t_{ji}\frac{\partial}{\partial y_j}$. Thus, for $w \in \mathfrak{Exp}(\mathbb{R}^n,\mathbb{R})$ we get

$$\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix} w(T\mathbf{x}) = T^* \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix} w(\mathbf{y}) = T^{\mathrm{T}} \begin{bmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{bmatrix} w(\mathbf{y}). \tag{2.1}$$

For ease of explanation and to avoid cumbersome notation we use ∂_x and ∂_y to denote the n-tuples of partial derivatives $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\}$ and $\{\frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_n}\}$, respectively.

LEMMA 10. *Let $T \in \mathbb{R}^{n\times n}$ define an invertible linear change of coordinates of $\mathbb{R}^n$ by $\mathbf{x} \mapsto T\mathbf{x} =: \mathbf{y}$. Then T induces an $\mathbb{R}$-algebra isomorphism $\psi : \mathbb{R}[\partial_x] \longrightarrow \mathbb{R}[\partial_y]$ by the linear change of variables $\partial_x \mapsto T^{\mathrm{T}}\partial_y$. Suppose $\mathcal{I} \subseteq \mathbb{R}[\partial_x]$ is an ideal, then $\psi(\mathcal{I})$ is an ideal in $\mathbb{R}[\partial_y]$. Consider the following two behaviors*

$$\mathfrak{B}_x := \{w(\mathbf{x}) \in \mathfrak{Exp}(\mathbb{R}^n, \mathbb{R}) \mid m(\partial_x)w = 0 \ \ \textit{for all } m \in \mathcal{I}\},$$
$$\mathfrak{B}_y := \{w(\mathbf{y}) \in \mathfrak{Exp}(\mathbb{R}^n, \mathbb{R}) \mid m(\partial_y)w = 0 \ \ \textit{for all } m \in \psi(\mathcal{I})\}.$$

Let $v_y, v_x \in \mathbb{R}^n$ be related to each other by $v_y = Tv_x$. Then there is a bijective set map between $\mathfrak{B}_x|_{v_x}$ and $\mathfrak{B}_y|_{v_y}$. Proof : That ψ is an isomorphism of n-variable polynomial algebras is clear from the fact that T^{T} is non-singular. It then follows that $\psi(\mathcal{I})$ is an ideal of $\mathbb{R}[\partial_y]$. Now notice that equation (2.1), together with the fact that T is invertible, shows that there is a set bijection between $\mathfrak{B}_x$ and $\mathfrak{B}_y$ given by $\widetilde{\psi} : \mathfrak{B}_y \to \mathfrak{B}_x$ with $\widetilde{\psi}(w(\mathbf{y})) = w(T\mathbf{x})$. This follows from the following argument. First observe that for $w \in \mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$ we have from equation (2.1) $\partial_x w(T\mathbf{x}) = T^{\mathrm{T}}\partial_y w(\mathbf{y})$, *i.e.*, for $1 \leqslant i \leqslant n$, $\frac{\partial}{\partial x_i} w(T\mathbf{x}) = \psi(\frac{\partial}{\partial x_i})w(\mathbf{y})$. More generally, for $m(\partial_x) \in \mathbb{R}[\partial_x]$

$$m(\partial_x)w(T\mathbf{x}) = \psi(m)(\partial_y)w(\mathbf{y}) = m(T^{\mathrm{T}}\partial_y)w(\mathbf{y}).$$

Hence it follows that $w(\mathbf{y})$ is in the kernel of $\psi(m)(\partial_y)$ if and only if $\widetilde{\psi}(w(\mathbf{y})) = w(T\mathbf{x})$ is in the kernel of $m(\partial_x)$. Thus from the one-to-one correspondence between behaviors and ideals we get $w \in \mathfrak{B}_y$ if and only if $\widetilde{\psi}(w) \in \mathfrak{B}_x$.

For the restriction, observe now that for $w(\mathbf{y}) \in \mathfrak{B}_y$, we have $(\widetilde{\psi}(w))(v_x t) = w(Tv_x t) = w(v_y t) \in \mathfrak{B}_y|_{v_y}$. Thus $\widetilde{\psi}$ induces a set bijection between $\mathfrak{B}_x|_{v_x}$ and $\mathfrak{B}_y|_{v_y}$. $\square$

Proof of Theorem 9:

(1 $\Rightarrow$ 2): We prove this implication by contradiction. Suppose 2 is not true, *i.e.*, the intersection ideal $\mathcal{I}_v$ is nonzero. Consider the 1-D behavior $\mathfrak{B}_v$, defined in equation (1.2), corresponding to $\mathcal{I}_v$. Since $\mathcal{I}_v \neq \{0\}$, it follows that $\mathfrak{B}_v$ is strictly contained in the set of all exponential 1-D trajectories. By Proposition 5 $\mathfrak{B}_v \supseteq \mathfrak{B}|_v$. Therefore, $\mathfrak{B}|_v \subseteq \mathfrak{B}_v \subsetneq \mathfrak{Exp}(\mathbb{R}, \mathbb{R})$, which contradicts the claim of 1.

(2 $\Leftrightarrow$ 3): This follows from the fact that ker $\varphi = \mathcal{I} \cap \mathbb{R}[\langle v, \partial\rangle] = \mathcal{I}_v$.

(3 $\Rightarrow$ 1): In order to prove this implication we will first prove a simpler case, and then we will make use of Lemma 10, which will render the general case into the simpler one.

Case 1 ($v = e_1 = \mathrm{col}[1, 0, \ldots, 0]$): The problem here reduces to proving $\varphi : \mathbb{R}[\partial_1] \to \mathbb{R}[\partial]/\mathcal{I}$ being injective implies e_1 is a free direction. We claim that φ being injective implies there exists a term ordering such that the standard monomials set Γ contains $\Gamma_1 := \{\nu \in \mathbb{N}^n \mid \nu = \lambda e_1, \lambda \in \mathbb{N}\}$. By Remark 8, it will then follow that e_1 is free. We take an elimination term ordering with $\partial_i \succ \partial_1$ for all $2 \leqslant i \leqslant n$. Then a Gröbner basis for $\mathcal{I}$, say $\mathcal{G}$, with this term ordering will have no element which has a monomial purely in ∂_1 as the leading monomial. For if $\mathcal{G}$ had a polynomial, say $f \in \mathbb{R}[\partial]$, with leading monomial purely in ∂_1, then since ∂_1 has least priority in the term ordering, the rest of the monomials in f will also be in ∂_1 only. Thus $f \in \mathbb{R}[\partial_1] \cap \mathcal{I} = \ker \varphi$, which contradicts our assumption that φ is injective. Now since $\mathcal{G}$ has no element with leading term purely in ∂_1, the initial ideal $\mathrm{in}_{\prec}(\mathcal{I})$, too, does not contain any monomial purely in ∂_1. In other words, the standard monomial set $\Gamma \supseteq \Gamma_1$.

Case 2 (general v): Note that since $v = \mathrm{col}[v_1, v_2, ..., v_n]$ is nonzero, one of its entries must be a nonzero real number. We may assume without loss of generality that $v_1 \neq 0$. For if it is not, then we can do a permutation on the variables $\{x_1, x_2, ..., x_n\}$ so that v changes to $\widetilde{v}$ and $\widetilde{v}_1 \neq 0$. Such a permutation exists because v has at least one entry nonzero. (By Lemma 10 it suffices to prove that $\widetilde{v}$ is free in this transformed system.) Now we define the following $(n \times n)$ real matrix and the linear transformation defined by it. Because v_1 has been assumed to be nonzero the following definition makes sense: $T := \begin{bmatrix} v_1 & 0 & \cdots & 0 \\ v_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ v_n & 0 & \cdots & 1 \end{bmatrix}^{-1}$. Note that $T^{-1}e_1 = v$, *i.e.*, $e_1 = Tv$.

Also, as in Lemma 10, T induces the $\mathbb{R}$-algebra isomorphism $\psi : \mathbb{R}[\partial_x] \ni \partial_x \mapsto T^{\mathrm{T}}\partial_y \in \mathbb{R}[\partial_y]$.

Now, by Lemma 10, it is enough to prove that $e_1 = Tv$ is a free direction in the autonomous system defined by the ideal $\psi(\mathcal{I})$. We claim that $\mathbb{R}[\frac{\partial}{\partial y_1}]$ injects into $\mathbb{R}[\partial_y]/\psi(\mathcal{I})$. Note that a feature of the T matrix is $\psi(\langle v, \partial_x\rangle) = \frac{\partial}{\partial y_1}$. Because of this we get the following commutative diagram.

$$\begin{array}{ccccc} \mathbb{R}[\langle v, \partial\rangle] & \hookrightarrow & \mathbb{R}[\partial_x] & \twoheadrightarrow & \mathbb{R}[\partial_x]/\mathcal{I} \\ \psi \downarrow\approx & & \psi \downarrow\approx & & \psi \downarrow\approx \\ \mathbb{R}[\frac{\partial}{\partial y_1}] & \hookrightarrow & \mathbb{R}[\partial_y] & \twoheadrightarrow & \mathbb{R}[\partial_y]/\psi(\mathcal{I}). \end{array}$$

It follows that $\mathbb{R}[\frac{\partial}{\partial y_1}]$ injects into $\mathbb{R}[\partial_y]/\psi(\mathcal{I})$. Thus we have reduced the general case to that of case 1, and thus the proof is complete. □

An immediate corollary to the above result is that no direction in a strongly autonomous behavior[1] is free. This is because, for a strongly autonomous system, $\mathcal{I}_v \neq \{0\}$ for all $v \neq 0$ (see [6]). By statement 2 of Theorem 9 it follows that in this case no direction is free.

COROLLARY 11. *If $\mathfrak{B}$ is strongly autonomous, then no direction is a free direction.*

3. Concluding remarks. In this paper, we have investigated the restriction of scalar autonomous n-D systems to 1-D subspaces. We have shown that a given direction may turn out to be *free*: every possible 1-D trajectory can be obtained by restriction of trajectories in the original system. Then we gave a set of algebraic criteria equivalent to a given direction being free.

REFERENCES

[1] D. NAPP AVELLI AND P. ROCHA, *Autonomous multidimensional systems and their implementation by behavioral control*, Systems and Control Letters, 59 (2010), pp. 203 – 208.
[2] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators I*, Springer: Grundlehren der mathematischen Wissenschaften, 2 ed., 1990.
[3] U. OBERST, *Multidimensional constant linear systems*, Acta Appl. Math., 20 (1990), pp. 1–175.
[4] ———, *The constructive solution of linear systems of partial difference and differential equations with constant coefficients*, in Constructive Algebra and Systems Theory, B. Hanzon and M. Hazewinkel, eds., Royal Netherlands Academy of Arts and Sciences, 2006, pp. 205–233.

[1]Strongly autonomous means that $\mathbb{R}[\partial]/\mathcal{I}$ is an artinian ring, see [7].

[5] U. Oberst and F. Pauer, *The constructive colution of linear systems of partial difference and differential dquations with constant coefficients*, Multidimensional Systems and Signal Processing, Kluwer Academic Publishers, 12 (2001), pp. 253–308.

[6] D. Pal and H. K. Pillai, *On restrictions of n-d systems to* 1-*d subspaces*, in Proceedings of the 7th International Workshop on Multidimensional (nD) Systems, Poitiers, France, September 2011.

[7] H. K. Pillai and S. Shankar, *A behavioral approach to control of distributed systems*, SIAM Journal on Control and Optimization, 37 (1998), pp. 388–408.

[8] J. C. Willems, *Paradigms and puzzles in the theory of dynamical systems*, IEEE Tansactions on Automatic Control, 36 (1991), pp. 259–294.