

# THE BOTTLENECK BICONNECTED STEINER NETWORK PROBLEM\*

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**Abstract.** We present the first exact algorithm for constructing minimum bottleneck 2-connected Steiner networks containing at most  $k$  Steiner points, where  $k > 2$  is a constant integer. Given a set of  $n$  terminals embedded in the Euclidean plane, the objective of the problem is to find the locations of the Steiner points, and the topology of a 2-connected graph  $N_k$  spanning the Steiner points and the terminals, such that the length of the bottleneck (the longest edge of  $N_k$ ) is minimised. The problem is motivated by the modelling of relay-augmentation for optimisation of energy consumption in survivable wireless communication networks. Our algorithm employs Voronoi diagrams and properties of block cut-vertex decompositions of graphs to find an optimal solution in  $O(h(k)n^k \log^{5k-1} n)$  steps, where  $h(k)$  is a function of  $k$  only.

**Key words.** Survivable Steiner network, bottleneck optimisation, 2-connected, block cut-vertex decompositions, Voronoi diagrams

**AMS subject classifications.** 05C40, 05C85, 90B10, 90B18, 68M10, 90B85

**1. Introduction and preliminaries.** We study the problem of constructing optimal *bottleneck biconnected Steiner networks* in the Euclidean plane. Besides [5] and papers on constructing so called *bottleneck biconnected spanning subgraphs* (see [6, 8]), few results in the literature are directly relevant to this problem. However, much progress has been made in the field of *bottleneck Steiner trees* (that is, when the solution is simply required to be connected). This problem has been shown to be NP-hard in the Euclidean and rectilinear planes, and also in general graphs (see [2, 9]); and recently Bae et al. in [1] and Brazil et al. in [3] produced exact algorithms that run in polynomial time for constant  $k$ .

We begin by presenting a few preliminary results, more detail of which can be found in [5]. For any graph  $G$  we denote the longest edge of  $G$  by  $e_{\max}(G)$  and its length by  $\ell_{\max}(G)$ . Let  $X$  be a set of vertices (called *terminals*) embedded in  $\mathbb{R}^2$ . The Euclidean *bottleneck 2-connected  $k$ -Steiner network problem* requires one to construct a 2-connected network  $N_k$  spanning  $X$  and a set  $S_{k'}$  of  $k' \leq k$  Steiner points, such that  $\ell_{\max}(N_k)$  is a minimum across all such networks. The variables are  $k'$ , the set  $S_{k'} \subset \mathbb{R}^2$ , and the topology of the network. An optimal solution to the problem is called a *minimum bottleneck 2-connected  $k$ -Steiner network*, or  $(2, k)$ -MBSN.

A *block* of  $G$  is a maximal 2-connected subgraph. Let  $\mathcal{Y}(G)$  be the set of blocks of  $G$ . The well-known *block cut forest* (BCF) of  $G$  is a forest  $F_{\mathcal{Y}(G)}$  with  $V(F_{\mathcal{Y}(G)}) = \{Y_i \in \mathcal{Y}(G)\} \cup \{z_i : z_i \text{ is a cut vertex of } G\}$  and  $E(F_{\mathcal{Y}(G)}) = \{Y_i z_j : z_j \in Y_i\}$ . For any subgraph  $F$  of  $F_{\mathcal{Y}(G)}$  the corresponding (vertex-induced) subgraph of  $G$  is the graph  $\widehat{F}$  where  $u \in V(\widehat{F})$  if and only if  $Y_i \in V(F)$  and  $u \in Y_i$ . By [7] the BCF of a graph with  $m$  edges can be constructed in time  $O(m)$ .

For every isolated component  $G_i$  of  $G$ , if  $G_i$  is a block then define  $b(G_i) := 2$ , else  $b(G_i) := |\mathcal{Y}_0(G_i)|$ . Finally, let  $b(G) := \sum b(G_i)$ . It is easy to show that if  $G_1$  is an edge subgraph of  $G_2$  then  $b(G_1) \geq b(G_2)$ . Any 2-connected graph  $Q$  containing  $G$  and at most  $k$  Steiner points is called a  *$k$ -block closure* of  $G$ . If  $\ell_{\max}(Q) \leq \ell_{\max}(Q')$

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for any  $k$ -block closure  $Q'$  of  $G$ , then  $Q$  is an *optimal*  $k$ -block closure of  $G$ . From [5] we know that for every leaf-block  $Y$  of  $G$  and any  $k$ -block closure of  $G$ , there exists at least one Steiner edge incident to a non-cut-vertex member of  $Y$ ; also, every isolated block  $W$  of  $G$  contains least two distinct Steiner edges incident to  $W$ .

Let  $R$  be the *2-relative neighbourhood graph*, or 2-RNG, on  $X$ . A 2-RNG on  $X$  can be constructed in time  $O(n^2)$  and has  $O(n)$  edges (see [6]). For any  $G$  we denote the edge-subgraph of  $G$  containing all edges of  $G$  of length at most  $r$  by  $G(r)$ . We assume that  $N_k$  satisfies the following proposition.

PROPOSITION 1.1 ([5]). *There exists a  $(2, k)$ -MBSN  $N_k$  on  $X$  such that  $N_k$  is a subgraph of  $R$  and the degree of  $v$  is at most 5 for every  $v \in V(N_k)$ .*

COROLLARY 1.2 ([5]). *Let  $G = R(\ell_{\max}(N_k - S_{k'}))$  and let  $G^+$  be any optimal  $k$ -block closure of  $G$ . Then  $G^+$  is a  $(2, k)$ -MBSN on  $X$ .*

Bae et al. in [1] provide new machinery for solving the *bottleneck  $k$ -Steiner tree problem* for fixed topologies, producing an algorithm of  $O(n^k)$  complexity. Since we may assume that  $N_k$  contains no Steiner point cycles, a minor reformulation makes their machinery suitable for the 2-connected case. The graph  $\mathcal{N}$  is called an *abstract topology* if and only if  $\mathcal{N}$  is full Steiner tree topology on  $a'' + k'$  vertices  $s_1, \dots, s_{k'}, X_1, \dots, X_{a''}$ , where  $k' \geq 2$ ,  $a'' \leq 5k'$ , each  $X_i$  is a labelled subset of  $X$ , and the degree of any Steiner point is at most 5 and no less than 2.

PROBLEM 1.3 (Fixed topology on subsets problem). *Suppose we are given a set  $X$  of  $n$  terminals in the plane, a positive integer  $k' \geq 2$ , and an abstract topology  $\mathcal{N}$ . Find a placement of the  $k'$  Steiner vertices to obtain a minimum bottleneck Steiner tree  $N^*$  that has the same topology as  $\mathcal{N}$ , such that if  $s_j$  is adjacent (in  $\mathcal{N}$ ) to  $X_i$  then  $s_j$  is adjacent (in  $N^*$ ) to a closest vertex of  $X_i$ .*

**2. Constructing abstract topologies.** By Corollary 1.2 one can construct a  $(2, k)$ -MBSN on  $X$  by finding an optimal  $k$ -block closure of the subgraph  $R(d)$  of the 2-RNG  $R$  on  $X$ , where  $d = \ell_{\max}(N_k - S_{k'})$ . In turn, one can construct an optimal  $k$ -block closure for  $G = R(d)$  by solving Problem 1.3 for every abstract topology  $\mathcal{N}$  on subsets of  $X$ , and choosing a cheapest solution for which  $G \cup N^*$  is 2-connected. The first step is to prune the number of potentially optimal terminal combinations for  $\mathcal{N}$ .

**2.1. Linked pairs.** Let  $\mathcal{H}$  be a minimum-cardinality partition of  $F_{\mathcal{Y}(G)}$  such that each member of  $\mathcal{H}$  is a path in  $F_{\mathcal{Y}(G)}$  with interior vertices of degree-two only. We may assume that each member  $H \in \mathcal{H}$  is of the form  $Y_1, \dots, Y_{q_H}$  for some  $q_H \geq 1$ , where each  $Y_i$  is a block. Clearly  $q_H = 1$  only if  $H$  is an isolated vertex of  $F_{\mathcal{Y}(G)}$ . Suppose that  $q_H \neq 1$ . For each  $i \in \{2, \dots, q_H\}$  let  $\tau_i$  be the cut-vertex of  $G$  common to  $Y_i$  and  $Y_{i-1}$ . Let  $G_0$  be the component of  $G - \tau_2$  containing no vertices of  $Y_2$ , and let  $W_0 = G_0 - \text{int}(Y_1)$ . Let  $\tau_1 = W_0 \cap Y_1$  (note that  $\tau_1$  may contain more than one vertex) and let  $\text{int}(W_0) = W_0 - \tau_1$ . For every  $i \in \{1, \dots, q_H\}$  let  $W_i = Y_i - \tau_i$ . Finally, let  $G_1$  be the component of  $G - \tau_{q_H}$  containing no vertices of  $Y_1$ , let  $W_{q_H+1} = G_1 - Y_{q_H}$ , and let  $\tau_{q_H+1} = Y_{q_H} - \text{int}(Y_{q_H}) - \tau_{q_H}$ . Note that  $W_0$  or  $W_{q_H+1}$  may be empty.

Now consider any  $k$ -block closure  $G^k$  of  $G$ . We begin a recursive definition by setting  $t = 0$ ,  $\text{rg}(t) = 0$ ,  $H_t = H$ , and  $W_0^t = W_0$ . In general: let  $P_{t+1}$  be a path in  $G^k$  with no internal vertices in  $\widehat{H}$  connecting  $\text{int}(W_0^t)$  to  $W_{\text{rg}(t+1)}^t$  where  $\text{rg}(t+1) \in$

$\{\text{rg}(t) + 1, \dots, q_H + 1\}$  is chosen to be as large as possible. Let  $Y^{t+1} = Y_1 \cup \dots \cup Y_{\tilde{t}}$  where  $\tilde{t} = \min\{q_H, \text{rg}(t+1)\}$ , let  $W_0^{t+1} = Y^{t+1} \cup P_1 \cup \dots \cup P_{t+1}$ , and let  $H_{t+1}$  be the BCF of the graph  $\widehat{H}_{t+1} = \widehat{H} \cup P_1 \cup \dots \cup P_{t+1}$ .

If  $W_0$  is not empty then  $P_1$  exists lest a vertex in  $\tau_1$  be a cut-vertex of  $G^k$ . However, if  $Y_1$  is a leaf-block of  $G$  then  $W_0$  is empty and we set  $\text{rg}(1) = 1$  and  $P_1 = \emptyset$ . For general  $t > 0$  the path  $P_{t+1}$  exists in  $G^k$  unless  $\text{rg}(t) = q_H + 1$ . Let  $b$  be the value of  $t$  for which  $\text{rg}(t+1) = q_H + 1$ . Note that for any  $t < b - 1$  the end-edge of  $P_{t+1}$  incident to  $W_{\text{rg}(t+1)}$  is a Steiner edge.

**OBSERVATION 2.1.**  $W_0^{t+1}$  is 2-connected. Therefore  $H_{t+1}$  is a path and  $W_0^{t+1}$  is an end-block of  $\widehat{H}_{t+1}$ . In particular,  $\widehat{H}_{b+1}$  is 2-connected.

By the maximality of  $\text{rg}(t+1)$  it follows that  $P_{t+1}$  is internally disjoint from  $P_t$  for all  $t > 0$ . Therefore, since  $P_{t+1}$  is incident to the interior of  $W_0^t$ ,  $P_{t+1}$  has an end-vertex in some  $W_{\text{lf}(t)}$  where  $\text{rg}(t-1) \leq \text{lf}(t) < \text{rg}(t)$ , or in  $\text{int}(Y_{\text{rg}(t)})$ . For  $0 < t \leq b$  let  $e_2^t$  be the Steiner edge contained in  $P_t$  and incident to  $W_{\text{rg}(t)}$  and let  $e_1^t$  be the Steiner edge contained in  $P_{t+1}$  and incident to  $W_{\text{lf}(t)}$ . Note that  $\text{lf}(t) = \text{rg}(t-1)$  only if  $e_1^t$  is incident to  $\tau_{\text{rg}(t-1)}$ . We refer to  $(e_1^t, e_2^t)$  as a *linked pair* of  $H$ . Each  $e_1^t$  is referred to as the *left* member of the linked pair, and  $e_2^t$  as the *right* member. Let  $E_H = \{(e_1^t, e_2^t)\}$  be the set of all linked pairs of  $H$  and observe that  $|E_H| = b$ . In Fig. 2.1 we illustrate the construction of  $(e_1^t, e_2^t)$ . In this representation we denote Steiner points by black-filled circles and Steiner edges by single lines. Blocks are represented by double circles, and edges of the BCF are depicted by double lines. The cut vertices of the BCF (i.e., the  $z_i$ ) are not depicted separately.

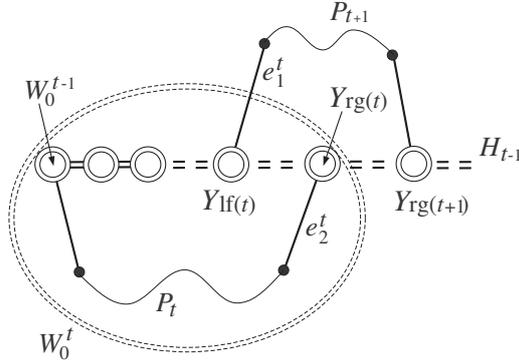


FIG. 2.1. *Linked pair  $(e_1^t, e_2^t)$  constructed at step  $t+1$*

**2.2. Viable neighbour-substitutions.** We first show that the connectivity of  $G^k$  essentially only depends on the left/right ordering of the edges in each linked pair. For any vertex  $x$  of  $\widehat{H}$  let  $\text{in}(x) = j$  if and only if  $W_j$  contains  $x$  for some  $j \in \{1, \dots, q_H\}$ ; we also similarly define  $\text{in}(e) = j$  when an end-vertex of  $e$  is contained in  $W_j$ . The *neighbour substitution* operation takes as input  $(e, x)$ , where  $e$  is a Steiner edge incident to  $\widehat{H}$  and  $x$  is a vertex of  $\widehat{H}$ , and replaces the current non-Steiner end-vertex of  $e$  by  $x$ .

Let  $\mu = \mu(H) = (e_1^1, e_2^1, \dots, e_1^b, e_2^b, f_1, \dots, f_{p(H)})$  where the  $e_i^j$  are the members of  $E_H$ , and  $E'_H = \{f_i : 1 \leq i \leq p(H)\}$  is the set of all Steiner edges incident to  $\widehat{H}$  that

do not occur in any member of  $E_H$ . Let  $\chi = \chi(H) = (\chi^1, \dots, \chi^{2b+p(H)})$  be a sequence of (not necessarily distinct) vertices of  $\widehat{H}$ . Let  $G^k(\chi)$  be the graph that results from  $G^k$  by simultaneously performing all neighbour substitutions contained in  $\{(\mu^i, \chi^i)\}$ , where  $\mu^i$  is the  $i$ -th term of  $\mu$ . We say that  $\chi$  is *viable* if and only if the following conditions hold:

- (1)  $\text{in}(\chi^2) \geq 1$ .
- (2)  $\text{in}(\chi^{i-1}) \leq \text{in}(\chi^i)$  and  $\chi^{i-1} \neq \tau_{\text{in}(\chi^i)+1}$  for any even  $i \leq 2b$ .
- (3)  $\chi^i$  is not a cut-vertex of  $G^k(\chi)$  for any  $i$  such that  $1 \leq i \leq 2b + p(H)$ .

PROPOSITION 2.2 ([4]). *If  $\chi$  is viable then  $G^k(\chi)$  is 2-connected.*

Our main algorithm contains a search procedure that iterates through viable sequences, and therefore we need to show that a sequence  $\chi$  can be verified as being viable in “reasonable” time. Under the assumption that preprocessing has been performed on  $G$ , specifically that the BCF of  $G$  has been specified, it should be clear that Conditions (1) and (2) of viability are verifiable in  $O(k)$  steps. We now show that Condition (3) can also be verified in a total number of steps bounded above by a function of  $k$  only.

Let  $\chi_1$  be any neighbour-substitution sequence that satisfies Conditions (1) and (2) for viability, but not Condition (3). Then any minimal edge-cut of  $G^k(\chi_1)$  may only contain Steiner edges, or edges incident to  $\tau_1$  or  $\tau_{q_H}$  and not contained in  $E(\widehat{H})$ . Therefore for every Steiner edge  $e \in E(G^k)$  incident to  $\widehat{H}$  there exists a unique set  $\beta(e)$  such that either  $\beta(e) = \emptyset$  or  $e \in \beta(e)$  and, for any sequence  $\chi_1$  satisfying Conditions (1) and (2) but such that the endpoints of all edges in  $\beta(e)$  coincide, Condition (3) is violated and  $\beta(e)$  is a minimal edge-cut of  $G^k(\chi_1)$ . In the main algorithm we only consider  $G$  and  $\mathcal{N}$  with  $b(G) \leq 5k$  and  $|E(\mathcal{N})| \in O(k)$ . Therefore  $|\mathcal{H}| \in O(k)$  and all  $\beta(e)$  (for all  $H \in \mathcal{H}$ ) can be constructed in the required time; hence Condition (3) can also be verified within this time for any neighbour-substitution sequence.

**2.3. Constructing an abstract topology by means of linked pairs.** In this subsection we show how the choice of linked pairs and the topology,  $\mathcal{N}$ , induced by the Steiner edges of  $G^k$  may be used to construct an abstract topology. Let  $E_S = \{e_j\}_{j \in I_S}$ , for some index set  $I_S$ , be the set of external Steiner edges (i.e., Steiner edges incident to terminals) of  $G^k$ . Let the *marker* of any linked edge  $e_j^i$  of  $G^k$  initially be defined as  $\text{mk}(e_j^i) := \text{in}(e_j^i)$ , and let  $\mathcal{M} = \{\text{mk}(e) : e \text{ is a linked edge of } G^k\}$ . For any  $e \in E_S$  incident to some  $\widehat{H}(e)$ , where  $H(e) \in \mathcal{H}$  and  $H(e) = Y_1, \dots, Y_{q_{H(e)}}$ , we define the *colour-set* of  $e$  with respect to  $\mathcal{M}$  as

$$J(e, \mathcal{M}) = \begin{cases} Y_1 \cup \dots \cup Y_{\text{mk}(e)} - \tau_{\text{mk}(e)+1} & \text{if } e \text{ is a left linked edge} \\ W_{\text{mk}(e)} \cup \dots \cup W_{q_{H(e)}} & \text{if } e \text{ is a right linked edge} \\ Y_1 \cup \dots \cup Y_{q_{H(e)}} & \text{for all other } e \end{cases}$$

For any  $G'$  let  $S(G')$  be the subgraph of  $G'$  induced by the Steiner edges. Let  $\mathcal{G}$  be the set of all  $k$ -block closures of  $G$  such that for any  $G' \in \mathcal{G}$  the graph  $S(G')$  is isomorphic to  $\mathcal{N}$ , and for any  $j \in E_S$  edge  $e_j$  is incident to a vertex of  $J(e_j, \mathcal{M})$  in  $G'$ . Observe that  $G^k \in \mathcal{G}$ . We begin by letting the terminals of  $\mathcal{N}$  be the members of  $\{J(e_j, \mathcal{M})\}_{j \in I_S}$ ; specifically, for every  $j \in I_S$  the terminal incident to edge  $e_j$  is  $J(e_j, \mathcal{M})$ . If  $e_j$  and  $e_i$  are distinct members of  $E_S$  then  $J(e_j, \mathcal{M})$  and  $J(e_i, \mathcal{M})$  are considered as distinct terminals of  $\mathcal{N}$ . We may assume that  $\mathcal{N}$  is a forest with no internal terminals.

Let  $N^*$  be a solution to the fixed topology on subsets problem for  $\mathcal{N}$ . Note that  $\ell_{\max}(G \cup N^*) \leq \min\{\ell_{\max}(G') : G' \in \mathcal{G}\}$ , but that  $G \cup N^*$  is not necessarily 2-connected. However, the next lemma shows that there exists a set  $\Gamma = \{\gamma(e_j) : j \in I_S\}$ , where each  $\gamma(e_j)$  is a (possibly empty) subset of  $J(e_j, \mathcal{M})$ , such that if the terminal-set of  $\mathcal{N}$  is modified to  $\{J(e_j, \mathcal{M}) - \gamma(e_j)\}_{j \in I_S}$  then  $G \cup N^*$  will be an optimal member of  $\mathcal{G}$ . We denote this updated abstract topology by  $\mathcal{N}(\Gamma)$ , and its optimal embedding by  $N^*(\Gamma)$ .

LEMMA 2.3 ([4]). *There exists a set  $\Gamma = \{\gamma(e_j)\}_{j \in I_S}$ , where  $\gamma(e_j) \subset J(e_j, \mathcal{M})$ , such that  $G \cup N^*(\Gamma)$  is an optimal member of  $\mathcal{G}$ .*

Let  $G(\mathcal{M}, \mathcal{N})$  be an optimal member of  $\mathcal{G}$  produced by the method in the previous lemma.

PROPOSITION 2.4. *There exists an  $\mathcal{M}$  and an abstract topology  $\mathcal{N}$  on  $\{J(e_j, \mathcal{M}) - \gamma(e_j)\}_{j \in I_S}$  such that  $G(\mathcal{M}, \mathcal{N})$  is a  $(2, k)$ -MBSN on  $X$ .*

*Proof.* Let  $G^+$  be any optimal  $k$ -block closure of  $G = R(d)$ , where  $d = \ell_{\max}(\overline{N}_k)$ . Let  $\mathcal{M}$  be defined on a set of linked pairs of  $G^+$  and let  $\mathcal{N}$  be have the topology of  $S(G^+)$ . The proposition now follows from the previous lemma.  $\square$

**3. The algorithm.** In order to utilise Proposition 2.4 in our algorithm we need a procedure for constructing every potentially optimal  $\mathcal{N}$ . To specify the terminals of  $\mathcal{N}$  for a given  $\mathcal{M}$  we need the following: the set  $\mathcal{H}$ ; a unique  $H(e) \in \mathcal{H}$  for each external Steiner edge  $e$ , such that  $e$  is required to be incident to  $\widehat{H}(e)$  in  $G(\mathcal{M}, \mathcal{N})$ ; and the set of linked pairs  $E_H$  for every  $H \in \mathcal{H}$ . We encapsulate this information as follows: let  $\mathcal{N}_0$  be any abstract topology on  $\mathcal{H}$  and let  $\mathcal{E} = \{E_H : H \in \mathcal{H}\}$ . An *optimal embedding* of the structure  $(\mathcal{N}_0, \mathcal{E})$  is a cheapest  $G(\mathcal{M}, \mathcal{N})$  for all marker-sets  $\mathcal{M}$  and abstract topologies  $\mathcal{N}$  consistent with  $\mathcal{E}$  and  $\mathcal{N}_0$ . We now present the *binary marker-search* process, which acts recursively on the possible locations of the markers to find an optimal embedding of  $(\mathcal{N}_0, \mathcal{E})$ .

Let  $a = 1$  and for each linked Steiner edge  $e$  of  $\mathcal{E}$  let  $\text{mk}_a(e)$  be a median of  $I_{H(e)} = \{1, \dots, q_{H(e)}\}$ , and let  $\mathcal{M}_a$  be the set of these markers. For any Steiner point  $s$  of  $G(\mathcal{M}_a, \mathcal{N})$  we place an arbitrary ordering  $e^1, \dots, e^{p(s)}$ , where  $p(s) \leq \deg(s)$ , on the edges incident to  $s$  which are members of linked pairs. Initially we set  $j(s) = p(s)$ . Now let  $e_a = e_{\max}(G(\mathcal{M}_a, \mathcal{N}))$ . If  $e_a$  is not a member of a linked-pair then  $G(\mathcal{M}_a, \mathcal{N})$  is an optimal embedding of  $(\mathcal{N}_0, \mathcal{E})$ . Otherwise suppose that  $e_a \in \{e^1, \dots, e^{p(s)}\}$  for some  $s$ . For simplicity we assume throughout that  $e^{j(s)}$  is a left edge.

Let  $a = 2$  and let  $\text{mk}_a(e^{j(s)})$  be a median value of  $\{\text{mk}_{a-1}(e^{j(s)}), \dots, q_{H(e^{j(s)})}\}$ ; all other markers remain at their current positions. Reconstruct  $G(\mathcal{M}_a, \mathcal{N})$  with respect to the new marker set  $\mathcal{M}_a$  and let  $e_a = e_{\max}(G(\mathcal{M}_a, \mathcal{N}))$ . Now suppose that we are at a general step  $a \geq 2$ , and suppose that  $e_a \in \{e^1, \dots, e^{p(s)}\}$  for some  $s$ . Let  $j(s)$  be the largest element of  $\{1, \dots, p(s)\}$  such that  $\text{mk}_a(e^{j(s)}) < q_{H(e^{j(s)})}$  (if no such  $j(s)$  exists then we have found an optimal embedding of  $(\mathcal{N}_0, \mathcal{E})$ ). We increment  $a$  by 1 and let  $\text{mk}_a(e^{j(s)})$  be a median of  $\{\text{mk}_{a-1}(e^{j(s)}), \dots, q_{H(e^{j(s)})}\}$ . Each  $\text{mk}_a(e^i)$ , where  $i > j(s)$ , is now moved back to a median of  $\{1, \dots, q_{H(e^i)}\}$  and  $G(\mathcal{M}_a, \mathcal{N})$  is constructed as before. We continue until no further improvement in  $\ell_{\max}(G(\mathcal{M}_a, \mathcal{N}))$  can be found.

Let  $G^{\text{mk}}$  be an optimal  $k$ -block closure of  $G$  produced by the binary marker-search process. Let  $r(G)$  be the length of a longest Steiner edge in  $G^{\text{mk}}$ . The following result is straightforward (a similar result appears in [5]).

LEMMA 3.1. *If  $G_1$  is an edge-subgraph of  $G_2$  then  $r(G_1) \geq r(G_2)$ .*

A *feasible* topology is 2-connected edge-linked abstract topology  $(\mathcal{N}_0, \mathcal{E})$  with  $k' \leq k$  Steiner points where the degree of any Steiner points is at most 5.

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Construct a  $(2, k)$ -MBSN

**Input:**  $X$  and a positive integer  $k$ .

**Output:** A  $(2, k)$ -MBSN on  $X$ .

1. Construct the 2-RNG  $R$  on  $X$  (*required time:*  $O(n^2)$ )
  2. Let  $L$  be the ordered set of edge-lengths occurring in  $R$ , where ties have been broken randomly
  3. Let  $d$  be a median of  $L$
  4. REPEAT (*number of times:*  $O(\ln n)$ )
  5.     Construct the BCF of  $G_d = R(d)$  (*required time:*  $O(n)$ )
  6.     IF  $b(G_d) > 5k$  exit loop and let  $d$  be median of next larger interval of  $L$
  7.     FOR all feasible topologies  $(\mathcal{N}_0, \mathcal{E})$  on  $G_d$
  8.         Run binary marker-search procedure on  $(\mathcal{N}_0, \mathcal{E})$
  9.         Let  $\mathcal{N}_{\text{opt}}$  be an optimal embedding of  $(\mathcal{N}_0, \mathcal{E})$
  10.        Let  $r(G_d)$  be the length of the longest Steiner edge of  $\mathcal{N}_{\text{opt}}$
  11.     IF  $r(G_d) \leq t$  let  $d$  be the next smaller median
  12.     ELSE let  $d$  be the next larger median
  13. UNTIL no smaller value of  $\max\{r(G_d), d\}$  can be found
  14. Output the instance with the minimum  $\max\{r(G_d), d\}$
- 

**THEOREM 3.2.** *Algorithm 1 correctly computes a  $k$ -MBSN in a time of  $O(n^k \log^{5k-1} n)$ .*

*Proof.* Let  $d_{\text{opt}} = \ell_{\max}(N_k - S_{k'})$  and  $G_{\text{opt}} = R(d_{\text{opt}})$ . Then by Corollary 1.2,  $G_{\text{opt}}^{\text{mk}}$  is a  $(2, k)$ -MBSN on  $X$ . Any  $d \in L$  such that  $b(G_d) \leq 5k$ , and  $G_d^{\text{mk}}$  is a  $(2, k)$ -MBSN on  $X$  is referred to as *valid*. Now let  $d \in L$  be some value considered in the binary search. If  $b(G_d) > 5k$  then there exists a valid  $d'$  such that  $d' > d$ . If  $r(G_d) \leq d$  then clearly there exists a valid  $d'$  such that  $d' \leq d$ , and if  $r(G_d) > d$  then, by Lemma 3.1, there exists a valid  $d'$  such that  $d' \geq t$ . Therefore a valid  $d'$  will be located by the binary search by decreasing  $d$  if  $r(G_d) \leq d$  and  $G_d$  is connected, and increasing  $d$  otherwise. Note that Line 7 requires constant time for fixed  $k$  since  $(\mathcal{N}_0, \mathcal{E})$  has structural complexity of  $O(k)$ . The binary marker-search procedure in Line 8 performs at most  $O(\ln^{5k-2} n)$  main iterations, since a binary search is performed on at most  $5k - 2$  Steiner edges. Therefore we require  $O(n^k \ln^{5k-2} n)$  steps for the binary marker-search in total. The result follows.  $\square$

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