An On-Line MUSIC Algorithm with Applications to Sparse Signal Reconstruction.

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1 Introduction

Sparse models are a class of systems in which prior knowledge indicates the presence of a majority of zero valued parameters in a model but not the location of the zeros in the parameter vector [1, 2, 3]. Such models occur in a wide range of applications [4, 5, 6, 7, 8, 3].

Many algorithms have been developed for estimating the location of the null parameters. Well known algorithms include, LASSO and Dantzig [9, 10]. These can be thought of as providing a form of regularization. It has been shown that, under appropriate conditions, these algorithms can find the correct location of the null parameters.

Another algorithm that has been widely used for sparse estimation problems is the MUSIC algorithm [11]. This algorithm was originally developed for multiple emitter locations in signal processing [11, 12]. It has also been applied to other sparse problems including blind multiband signal reconstruction [13, 3].

All of the algorithms mentioned above are "off-line" in the sense that one first collects a large set of data and then one applies the algorithm. However, in some applications, e.g. cognitive radio, it would be helpful if one could have an "online" version of the algorithm in which the estimates of the sparse model evolves as the data is collected. Thus, at any given time, one has an estimate of the sparse model.

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The estimated model improves as more data is collected.

The main contribution of the current paper is to develop an on-line form of the MUSIC algorithm. We test the algorithm by applying it to a particular sparse problem, namely, blind multiband signal reconstruction.

2 A Class of Sparse Estimation Problems

We consider a sparse estimation problem of the following form.

$$\underline{\widetilde{y}}[k] = A\underline{\widetilde{x}}[k] \tag{1}$$

where $\tilde{x}[k] \in \mathbb{C}^L$ denotes a vector of parameters, $\tilde{y}[k] \in \mathbb{R}^p$ denotes a given set of data and $A \in \mathbb{C}^{p\times L}$ represents a regression matrix.

We assume that prior knowledge indicates that only $2N < p$ of the elements of $\tilde{x}[k]$ are non-zero. We can then rewrite (1) as

$$\underline{\widetilde{y}}[k] = A(\mathbf{b}^*)\underline{\widetilde{x}}_{\mathbf{b}^*}[k] \tag{2}$$

where $A(\mathbf{b}^*) = \begin{bmatrix} a(\beta_1^*) & a(\beta_2^*) & \cdots & a(\beta_{2N}^*) \end{bmatrix}$, $\mathbf{b}^* = [\beta_1^*, \ldots, \beta_{2N}^*]^T$, β_i^* for $i = 1, 2, \ldots, 2N$ are the indices of the columns of A related to the non-zero elements $\underline{\widetilde{x}}[k]$, and the vector $\underline{\widetilde{x}}_{\mathbf{b}^*}[k]$ contains the non-zero elements of $\underline{\widetilde{x}}[k]$. We also consider that $a(\beta_i)^H a(\beta_j) = 0$, $i \neq j$.

Th goal is to find the location of the non-zero elements, i.e. $\mathbf{b}^*$ and then estimate $\underline{\widetilde{x}}_b$ which is a straightforward least squares problem .

3 The MUSIC Algorithm

The problem described in Section 2 can be solved using the MUSIC algorithm [11]. A brief description of the MUSIC algorithm is given below.

First, we define the covariance matrix $R = \sum_k \underline{\widetilde{y}}[k]\underline{\widetilde{y}}[k]^T$, and consider the problem (2). Then we have that

$$R = A\sum_k \underline{\widetilde{x}}[k]\underline{\widetilde{x}}[k]^H A^H = A(\mathbf{b}^*)\sum_k \underline{\widetilde{x}}_{\mathbf{b}^*}[k]\underline{\widetilde{x}}_{\mathbf{b}^*}[k]^H A(\mathbf{b}^*)^H \tag{3}$$

Moreover, if $\sum_k \underline{\widetilde{x}}_{\mathbf{b}^*}[k]\underline{\widetilde{x}}_{\mathbf{b}^*}[k]^H$ has full rank, then R will be singular since $2N < p$. Specifically, R will have at least $p-2N$ eigenvalues equal to zero. The eigenvectors associated with the zero valued eigenvalues form the Null space of R. Note that every vector $\underline{e}_m$ in the Null space of R satisfies the condition $A(\mathbf{b}^*)^H\underline{e}_m = 0$.

Let E be the $p \times (p-2N)$ matrix whose columns are the eigenvectors that form the Null space, then an estimate $\hat{\beta}_i$ of β_i^* for $i = 1, 2, \ldots, 2N$, can be found by determining the location of the largest $2N$ values of

$$P_{MU}(\beta_i) = \frac{1}{a(\beta_i)^H E E^T a(\beta_i)} \tag{4}$$

The corresponding $2N$ values of β are then assembled to form an estimate, $\widehat{\mathbf{b}}$, of $\mathbf{b}^*$.

Once $\widehat{\mathbf{b}}$ has been evaluated, then the correspondly estimate for $\underline{\widetilde{x}}_{\mathbf{b}}[k]$ can be calculated by a number of means. For example, the least squares estimate is

$$\underline{\widetilde{x}}_{\mathbf{b}}[k] = A(\mathbf{b})^{+}\underline{\widetilde{y}}[k] \tag{5}$$

where $^+$ denotes the Moore-Penrose pseudoinverse [14].

4 On-Line MUSIC Algorithm

As outlined in Section 3, a key step in the MUSIC algorithm is to find the Null subspace of R through the estimation of its eigenvectors and eigenvalues. Typically this step is performed off-line [11] after collecting a large amount of data. Here we develop an on-line version of the algorithm.

Let $R_k = \sum_{t=1}^{k} \underline{\widetilde{y}}[t]\underline{\widetilde{y}}[t]^H$ be the estimate of R at time k. Then, clearly,

$$R_{k+1} = R_k + \underline{\widetilde{y}}[k+1]\ \underline{\widetilde{y}}[k+1]^H \tag{6}$$

Then, an on-line estimate of the Null space of R can be achieved as follows:

Let $u_k^{(i)},\ i = 1, \ldots, p$ denote the estimate of the normalized eigenvector associated with the i-th largest eigenvalue of R.

At time k we have $\{u_k^{(i)}, P_k^{(i)}\},\ i = 1, \ldots, p$, and considering $P_k^{(0)} = I$

$$\tilde{u}_{k+1}^{(i)} = P_k^{(i-1)}u_k^{(i)} + \alpha_k P_k^{(i-1)} R_{k+1} P_k^{(i-1)} u_k^{(i)},\ \ \alpha_k > 0 \tag{7}$$

$$u_{k+1}^{(i)} = \tilde{u}_{k+1}^{(i)}/||\tilde{u}_{k+1}^{(i)}||, \tag{8}$$

when $||u_{k+1}^{(i)}|| \neq 0$, if $||u_{k+1}^{(i)}|| = 0$ we reset $u_k^{(i)}$ to some other value and repeat (7). Also, with $P_{k+1}^{(0)} = I$ we calculate

$$V_{k+1}^{(i)} = \begin{bmatrix} u_{k+1}^{(1)} & P_{k+1}^{(1)}u_{k+1}^{(2)} & \cdots & P_{k+1}^{(i-1)}u_{k+1}^{(i)} \end{bmatrix} \tag{9}$$

$$P_{k+1}^{(i)} = I - V_{k+1}^{(i)}V_{k+1}^{(i)+} \tag{10}$$

The estimate of E, the nullspace generating matrix for R is then

$$\hat{E}_{k+1} = \begin{bmatrix} u_{k+1}^{(2N+1)}, & \ldots, & u_{k+1}^{(p)} \end{bmatrix} \tag{11}$$

Note that the eigenvalues can be recursively estimated (if desired), via

$$\lambda_{k+1}^{(i)} = \lambda_k^{(i)} - \alpha_k(\lambda_k^{(i)} - u_k^{(i)T}R_{k+1}u_k^{(i)}).$$

The core ingredients of the above algorithm are equations (7) and (8). These equations are a form of stochastic approximation [15]. The use of stochastic approximation for the general problem of eigenvalue estimation has been explored in [16]. Our key contribution is to adopt this idea to develop an on-line MUSIC algorithm.

5 Application to Multiband Signal Reconstruction

We illustrate the performance of the on-line MUSIC algorithm by applying it to the problem of blind multiband signal reconstruction.

The problem of blind multiband signal reconstruction has been studied by previous researches, see [1, 2, 17, 3]. Without exception, these methods use off-line data analysis. Algorithms that fall into this category include Orthogonal Matching Pursuit, Basis Pursuit, Lasso, and Dantzig, among others. Also, by exploiting links to other related problems, e.g. multiple emitter location, other algorithms are possible [3], e.g the off-line MUSIC algorithm [11]. Our goal here is to utilize the on-line algorithm of Section 4.

The class of signals of interest in multiband signal reconstruction ([1, 2, 17, 3]) is:

Definition 1. *Let $\mathcal{M}$ be the set of continuous real signals $x(t)$ with associated Fourier transform, $X(f)$, such that $x(t)$ and $X(f)$ satisfy the assumptions that $x(t)$ is bandlimited to the frequency range $\mathcal{F} = [-\frac{1}{2T}, \frac{1}{2T}]$, $X(f)$ is restricted to have support no more than N disjoint bands in $\mathcal{F}$, and the width of each band does not exceed B.*

The number of non-zero bands is assumed known but the positions of the N bands of $X(f)$ are unknown. Also, it is that the support of $X(f)$ is sparse, i.e. the N disjoint bands occupy a small part $\mathcal{F}$.

5.1 Multi coset sampling

Here we describe a special kind of sampling (called multi coset sampling) which exploits the sparse nature of the data to yield a sampling rate below the Nyquist rate (see e.g [18, 13, 19]).

In multi coset sampling, samples are obtained by the selection of samples from an uniform grid of samples. In every block of L uniform samples, all samples are discarded except p samples that are specified by a set $\mathbf{C} = \{c_i\}_{i=1}^{p}$ (referred to as a sampling pattern) where $0 \leq c_1 < c_2 < \cdots < c_p < L-1$. The sampling sequence for $1 \leq i \leq p$ is defined by

$$x_{c_i}[k] = \begin{cases} x(kT) & k = mL + c_i, m \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

Passing these sequences through a lowpass filter given by, $h_{LP} = 1$ if $f \in \mathcal{F}_0$, and $h_{LP} = 0$ if $f \notin \mathcal{F}_0$. Where $\mathcal{F}_0 = [-\frac{1}{2LT}, \frac{1}{2LT}]$ for odd L, and $\mathcal{F}_0 = [0, \frac{1}{LT}]$ for even L. We denote the filtered signal by $\widetilde{x}_{c_i}[k] = x_{c_i}[k] * h_{LP}[k]$. Next, using the Fourier Transform we observe that

$$\begin{aligned} \widetilde{X}_{c_i}(f) &= \sum_{k=-\infty}^{\infty} \widetilde{x}_{c_i}[k] \cdot \exp(-j2\pi fTk), \\ &= \frac{1}{LT} \sum_{r=1}^{L} \exp(j2\pi \tfrac{\mathcal{K}_r}{L} c_i) \cdot \widetilde{X}_r(f) \qquad f \in \mathcal{F} \end{aligned} \tag{13}$$

where $\widetilde{X}_r(f) = X(f + \frac{\mathcal{K}_r}{LT}) h_{LP}(f)$, $\mathcal{K} = \{-\frac{L-1}{2}, \ldots, \frac{L-1}{2}\}$ for odd L, and $\mathcal{K} = \{-\frac{L}{2}, \ldots, \frac{L}{2} - 1\}$ for even L. Also, $\mathcal{K}_m$ denotes the m-th element in $\mathcal{K}$. We note that because of the multiband character of $x[k]$, $X(f + \frac{\mathcal{K}_r}{LT}) = 0$ for some of the $r \in \{1, \ldots, L\}$ for all $f \in \mathcal{F}_0$. Hence, $\widetilde{X}_r(f) = 0$, $f \in \mathcal{F}$, for the corresponding r. So, rewriting (13) in the time domain we have

$$\widetilde{x}_{c_i}[k] = \frac{1}{LT}\sum_{r=1}^{L}\exp(j2\pi\tfrac{\mathcal{K}_r}{L}c_i)\cdot\widetilde{x}_r[k] \qquad i=1,\ldots,p \tag{14}$$

or, in vector form

$$\underline{\widetilde{y}}[k] = A\underline{\widetilde{x}}[k] \tag{15}$$

where

$$\underline{\widetilde{y}}[k] = \begin{bmatrix}\widetilde{x}_{c_1}[k]\\ \vdots \\ \widetilde{x}_{c_p}[k]\end{bmatrix}; \qquad \underline{\widetilde{x}}[k] = \begin{bmatrix}\widetilde{x}_1[k]\\ \vdots \\ \widetilde{x}_L[k]\end{bmatrix}; \qquad A_{i,m} = \frac{1}{LT}\exp\left(j\tfrac{2\pi}{L}c_i\mathcal{K}_m\right).$$

Note in (15), that $p < L$, so the set of equations is underdetermined. However, the choice of $L \leq \frac{1}{BT}$ and the fact that the support of $X(f)$ is sparse, gives additional information. In particular, for at least $L-2N$ values of $r \in \{1,\ldots,L\}$, $X(f+\frac{\mathcal{K}_r}{LT})$ is zero for all $f \in \mathcal{F}_0$. Hence, the corresponding $\widetilde{X}_r(f)$ are zero for all $f \in \mathcal{F}$ and so, $\widetilde{x}_r[k] = 0$ for all k. The vector $\underline{\widetilde{x}}[k]$ is referred in the literature as being $2N$-sparse (formally, a vector $\mathbf{v}$ is called $2N$-sparse if the number of nonzero values in $\mathbf{v}$ is $2N$ [19]).

5.2 Blind On-Line Reconstruction of Multiband Signals

We next apply the on-line MUSIC algorithm of Section 4 to estimate $\mathbf{b}^*$ the vector containing the indicies of the non-empty frequency slots of the multiband signal. The blind multiband signal reconstruction problem can be cast into the general on-line MUSIC problem if we choose $A(\mathbf{b})$ and $a(\beta_i)$ as follows:

$$a(\beta_i) = \frac{1}{LT}\left[\exp\left(j\tfrac{2\pi}{L}c_1\beta_i\right) \quad \exp\left(j\tfrac{2\pi}{L}c_2\beta_i\right) \quad \ldots \quad \exp\left(j\tfrac{2\pi}{L}c_p\beta_i\right)\right]^T \in \mathbb{C}^p$$

From the definition of the entries of $\underline{\widetilde{x}}[k]$ in (13) we readily observe that when shifted in the frequency domain and summed, the original signal $x[k]$ is reconstructed. Namely, in the time domain, this corresponds to

$$x[k] = \sum_{r=1}^{L}\exp\{j2\pi\tfrac{\mathcal{K}_r}{L}k\}\underline{\widetilde{x}}_r[k] \tag{16}$$

$$= \sum_{i=1}^{2N}\exp\{j\frac{2\pi}{L}k\beta_i^*\}\underline{\widetilde{x}}_{\gamma_i^*}[k] \tag{17}$$

where γ_i are the index of the nonzero entries of $\underline{\widetilde{x}}[k]$, such that $\beta_i^* = \mathcal{K}_{\gamma_i^*}$, and $\exp\{j\frac{2\pi}{L}r\beta_i^*\}$ represent the corresponding shifts in the frequency domain.

Motivated by (17), and since we are given (at each time interval) an estimate $\widehat{\mathbf{b}} = \{\widehat{\beta}_1,\ldots,\widehat{\beta}_{2N}\} \subset \mathcal{K}$ of $\mathbf{b}^*$, we can readily reconstruct an estimate of the signal via

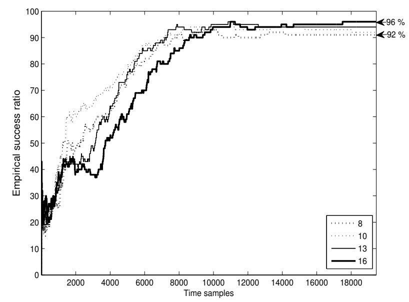
$$\widehat{x}[k] = \sum_{i=1}^{2N}\exp\{j\tfrac{2\pi}{L}k\widehat{\beta}_i\}\widetilde{x}_{\widehat{\gamma}_i}[k] \tag{18}$$

The full on-line algorithm is summarized in Table 1.

Table 1 . On-Line algorithm for blind reconstruction of multiband signals

1. Initialize: $\widehat{\mathbf{b}}$, $\{u_0^{(i)}, P_0^{(i)}\}$, $i = 1, \ldots, p$.
2. At each time k:
   (a) Update the eigenvector estimates, $u_k^{(i)}$, using (7)-(8).
   (b) Estimate $\widehat{\mathbf{b}}$ using MUSIC algorithm.
   (c) Estimate $\widehat{x}[k]$ using (18).

Figure 1 Empirical success ratio at each time, for several sampling pattern sizes, p.



6 Numerical Results

Here we present a numerical experiment to show the characteristics of the proposed algorithm.

Consider the class $\mathcal{M}$ of multiband signals, bandlimited in the frequency range $\mathcal{F} = [-10Ghz, 10GHz]$, with frequency support on $N = 2$ bands of width $B = 950MHz$. The signals are generated randomly by first drawing $\{a_i\}_{i=1}^{N/2}$ uniformly and at random in the frequency range $[0, 10GHz - B]$. Next, set $b = a_i + B$, for $1 \leq i \leq N/2$, and ensure that the bands do not overlap. Finally, generate $X(f)$ by $X(f) = d_i(A_R(f) + jA_I(f))$ if $f \in \bigcup_{i=1}^{N}[a_i, b_i]$ and $X(f) = 0$ otherwise . $A_R(f)$ and $A_I(f)$are drawn independently from a normal distribution with zero mean and unit variance. The constants d_i are chosen uniformly independently from the interval $[1, 5]$, for each band.

Multi coset sampling is used with $L = 19$, and where p goes from 8 to 16. The sampling pattern is chosen as $C = \{0, 1, \ldots, p-1\}$.

We run 100 tests, each one with a different signal. For each signal, we consider different sampling rates, with $p = 8, 10, 13, 16$. For each sampling rate we recover the signal using the proposed algorithm. To measure the performance of the algorithm, the empirical success ratio is calculated at each time as the ratio of simulations in which the signal $x[k]$ is successfully recovered. The results are shown in Figure 1. We see that for $p = 8$, we have estimated the correct frequency domain support of the signal in 80% of trials once $6,000$ data points have been analyzed. The success rate increases to above 90% as more data is analyzed

We also see from Figure 1 that higher sampling rates give a larger success ratio. However, for $p = 16$ the convergence is slower than for $p = 8$. This is because more eigenvectors must be estimated when $p = 16$.

7 Conclusions

A novel time-recursive algorithm for blind multiband signal reconstruction has been described. The performance of the proposed algorithm has been confirmed via a numerical example.

Bibliography

[1] M. Mishali and Y. Eldar, "Reduce and booset: Recovering arbitrary sets of jointly sparse vectors," *IEEE Trans. on Signal Proc.*, vol. 56, no. 10, pp. 4692–4702, October 2008.

[2] ——, "From theory to practice: Sub-Nyquist sampling of sparse wideband analog signals," *IEEE Journal of Selected Topics in Signal Proc.*, vol. 4, no. 2, pp. 375–391, April 2010.

[3] M. Rashidi, "Non-uniform sampling and reconstruction of multi-band signals and its application in wideband spectrum sensing of cognitive radio," *ArXiv e-prints*, October 2010.

[4] S. F. Cotter and B. D. Rao, "Sparse channel estimation via matching pursuit with application to equalization," *IEEE Trans. Comm*, pp. 374–377, 2002.

[5] C. Carbonelli, S. Vedantam, and U. Mitra, "Sparse channel estimation with zero tap detection," *Wireless Communications, IEEE Trans.*, vol. 6, no. 5, pp. 1743 –1763, may 2007.

[6] J. Homer, "Detection guided lms estimation of sparse channels," in *GLOBECOM 98. The Bridge to Global Integration. IEEE*, vol. 6, 1998, pp. 3704 –3709 vol.6.

[7] J. Homer, I. Mareels, R. Bitmead, B. Wahlberg, and A. Gustafsson, "Lms estimation via structural detection," *Signal Processing, IEEE Trans.*, vol. 46, no. 10, pp. 2651 –2663, oct 1998.

[8] W. Li and J. Preisig, "Estimation of rapidly time-varying sparse channels," *Oceanic Engineering, IEEE Journal of*, vol. 32, no. 4, pp. 927 –939, oct. 2007.

[9] R. Tibshirani, "Regression shrinkage and selection via the lasso," *Journal of the Royal Statistical Society, Series B*, vol. 58, no. 1, pp. 267–288, 1996.

[10] E. Candes and T. Tao, "The dantzig selector: Statistical estimation when p is much larger than n," *Annals of Statistics*, vol. 35, no. 6, pp. 2313–2351, 2007.

[11] R. Schmidt, "Multiple emitter location and signal parameter estimation," *Antennas and Propagation, IEEE Trans.*, vol. 34, no. 3, pp. 276 – 280, 1986.

[12] G. Golub and V. Pereyra, "Separable nonlinear least squares: the variable projection method and its applications," *Inverse Probl*, vol. 19, no. 2, pp. R1–R26, Jan 2003.

[13] P. Feng and Y. Bresler, "Spectrum-blind minimum-rate sampling and reconstruction of multiband signals," in *Acoustics, Speech, and Signal Processing, 1996. ICASSP-96. Conference Proceedings.*, vol. 3, may 1996, pp. 1688 –1691 vol. 3.

[14] D. Bernstein, *Matrix mathematics: theory, facts, and formulas.* Princeton University Press, 2009.

[15] H. Robbins and S. Monro, "A stochastic approximation method," *The Annals of Mathematical Statistics*, vol. 22, no. 3, pp. pp. 400–407, 1951.

[16] H.-F. Chen, H.-T. Fang, and L.-L. Zhang, "Recursive estimation for ordered eigenvectors of symmetric matrix with observation noise," *Journal of Mathematical Analysis and Applications*, Jan 2011.

[17] J. A. Tropp, J. N. Laska, M. F. Duarte, J. K. Romberg, and R. G. Baraniuk, "Beyond Nyquist: Efficient sampling of sparse bandlimited signals," *IEEE Transactions on Information Theory*, vol. 56, no. 1, pp. 520–544, January 2010.

[18] H. Landau, "Necessary density conditions for sampling and interpolation of certain entire functions," *Acta Math*, vol. 117, no. 7, pp. 1973–1984, Feb. 1967.

[19] M. Mishali and Y. Eldar, "Blind multiband signal reconstruction: Compressed sensing for analog signals," *IEEE Trans. on Signal Proc.*, vol. 57, no. 3, pp. 993–1009, March 2009.