A MODEL REDUCTION STRATEGY PRESERVING DISTURBANCE DECOUPLING PROPERTIES*

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Abstract. In this paper, we address the problem of model reduction with control relevant approximation criteria. We focus on a reduction strategy in which the feasibility to design a controller that completely decouples disturbances from to-be-controlled variables in a system is left invariant. The main result of this paper provides a model reduction strategy with the property that disturbance decoupling is possible for the full order system if and only if this is possible for the reduced order system. In addition, controllers that achieve disturbance decoupling for the reduced order system can be implemented on the full order system with guaranteed disturbance decoupling properties.

Key words. model reduction, disturbance decoupling, control relevant reduction

AMS subject classifications. 93A15, 93B27, 93B51, 93C05

1. Introduction and motivation.

The design of controllers for large scale dynamical systems usually involves complex (i.e., high order) models that contain all relevant physical phenomena in sufficient detail. The controller, when implemented in the control loop, is typically required to have short computation times and to be implementable with a numerically robust and reliable signal processor. The use of model order reduction techniques may become imperative to design such controllers on the basis of simplified models. There are two classical approaches to obtain low order controllers. This is achieved either by first approximating the model and inferring a low order controller based on this approximation, or by directly designing a controller for the complex model and by applying a reduction strategy to reduce the controller complexity. With common model reduction strategies, both approaches have the potential disadvantage to loose relevant information for the controller and to compromise performance in the reduction process. Obviously, in the question to derive low order controllers, it is of paramount interest to take the control objective into account and to judge the quality of the approximation on the basis of performance loss and possible loss of stability of the controlled system.

In a number of specific control synthesis problems it has been shown that it is possible to reduce a complex controller while providing performance guarantees on the closed loop behavior (see e.g. [3]). However, the synthesis of such a (complex) controller might be a difficult, and sometimes even an impossible, task. We therefore consider the question of control relevant model reduction in which the model of the to-be-controlled system is simplified while keeping the subsequent control system design into account. More specifically, we consider the question when the optimal closed loop performance can be kept invariant in such a reduction procedure. In [5] we have shown a number of results in which we demonstrated that with specific reduction strategies and with specific reduction orders, there is no need to compromise closed-loop performance in the sense that optimal controllers designed for the reduced order system still perform optimally when implemented on the original system. We address

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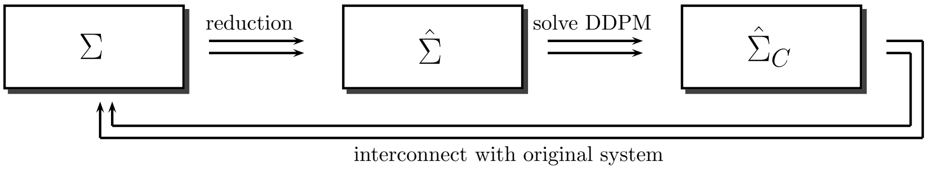

FIG. 2.1. *Model reduction strategy that keeps solvability of DDPM invariant.*

this problem for disturbance decoupling problems. These problems have been extensively considered in geometric control theory and have led to a solid understanding of the intricate state properties that lead to disturbance decoupling in a controlled system configuration. Many variations, including the problem of disturbance decoupled observer design and disturbance decoupling with additional stability requirements on the controlled system, have been addressed in this framework.

In this paper, we extend the results of [5] to the problem of disturbance decoupling where partial measurements of the state variable are available for the controller. This problem is referred to as the 'disturbance decoupling problem with measurement feedback' and is known by the acronym DDPM in geometric control theory. We will formalize this problem in the next section. The main result of this paper will be a model reduction strategy that keeps solvability of DDPM invariant. This means that we provide concise conditions under which solvability of DDPM in the full order system is equivalent to solvability of DDPM in the reduced order system. Moreover, a controller that decouples disturbances from to-be-controlled outputs for the reduced system will achieve this property when implemented on the original full order system and, conversely, a controller that achieves disturbance decoupling for the full order system will achieve disturbance decoupling for the reduced order system.

The outline of this paper is as follows. The main problem of this paper, the synthesis of a reduction strategy that keeps the solvability of DDPM invariant, is defined in Section 2. Section 3 includes notation and preliminaries that are relevant for the main results in Section 4. Conclusions are drawn, and recommendations for future research will be given in Section 5.

2. Disturbance decoupling with partial measurement feedback.

The problem to design a static state feedback law that achieves a disturbance signal to be completely decoupled from the output of a system has been widely discussed in geometric control theory. See, e.g. [1, 9]. In this paper, we focus on the problem where only partial measurements are available for the interconnection with a controller. See [7, 8]. Consider the linear time-invariant system Σ that is represented in state-space form by the equations:

$$\Sigma : \quad \begin{cases} \dot{x} = Ax + Bu + Gd, \\ y = Cx + Jd, \\ z = Hx + Du. \end{cases} \tag{2.1}$$

Here, $x(t) \in \mathbb{R}^n := \mathcal{X}$, $u(t) \in \mathbb{R}^u$, $d(t) \in \mathbb{R}^d$, $y(t) \in \mathbb{R}^y$ and $z(t) \in \mathbb{R}^z$ and the system matrices have appropriate dimensions. The disturbance decoupling problem with measurement feedback (DDPM) amounts to designing a (dynamic, linear, time-invariant) controller Σ_C that causally processes measurements y to control signals u

such that the interconnection of Σ with Σ_C results in a controlled system in which the disturbance d has no influence on the output z.

In this paper, the system (2.1) is assumed to be a complex system and the aim is to reduce its state dimension n. More specifically, we consider the problem of DDPM for a complex system as illustrated in Figure 2.1, and aim to find a simplified substitute model in which solvability of DDPM is preserved as an invariant property. Formally, this leads to the following problem:

Problem formulation: Given the nth order system Σ as in (2.1) and an integer $r < n$. Find an rth order approximate system $\hat{\Sigma}$ for Σ in such a way that any controller that solves DDPM for $\hat{\Sigma}$ will solve DDPM for Σ and, conversely, any controller that solves DDPM for Σ will solve DDPM for $\hat{\Sigma}$. In addition, characterize for which $r < n$ such a reduction will be possible.

3. Preliminaries and notation.

Consider the system Σ and let $x(t; x_0, u, d)$ denote the state trajectory of (2.1) corresponding to the input u, the disturbance d and the initial condition $x(0) = x_0$.

We call a subspace $\mathcal{V} \subset \mathcal{X}$ *controlled invariant*, or (A, B) invariant, if for all $x_0 \in \mathcal{V}$, there exists u such that $x(t; x_0, u, 0) \in \mathcal{V}$ for all $t \geq 0$. It is well known that the following statements are equivalent:

i. $\mathcal{V}$ is controlled invariant,
ii. $A\mathcal{V} \subset \mathcal{V} + \operatorname{im} B$, and
iii. there exists F such that $(A + BF)\mathcal{V} \subset \mathcal{V}$.

Let $\mathcal{F}(\mathcal{V})$ denote the set of matrices F that achieve property iii. It is easily seen that the property of controlled invariance is closed under addition. For the system (2.1), this means that there exists a well defined controlled invariant subspace $\mathcal{V}^*$ that contains any other controlled invariant subspace $\mathcal{V}$ such that, with $F \in \mathcal{F}(\mathcal{V})$, we have $(A + BF)\mathcal{V} \subset \mathcal{V}$ and $\mathcal{V} \subset \ker(H + DF)$.

Similarly, a subspace $\mathcal{S} \subset \mathcal{X}$ is called *conditioned invariant*, or (C, A) invariant if $A(\mathcal{S} \cap \ker C) \subset \mathcal{S}$. It is well known that $\mathcal{S}$ is conditioned invariant if and only if there exists L such that $(A + LC)\mathcal{S} \subset \mathcal{S}$.

Let $\mathcal{L}(\mathcal{S})$ denote the set of matrices L that satisfy the latter inclusion. Since the property of conditioned invariance is closed under intersection, there exists a well defined conditioned invariant subspace $\mathcal{S}^*$ that is contained in any other conditioned invariant subspace $\mathcal{S}$ such that, with $L \in \mathcal{L}(\mathcal{S})$, we have that $\operatorname{im}(G + LJ) \subset \mathcal{S}$ and $(A + LC)\mathcal{S} \subset \mathcal{S}$.

We call $\mathcal{S}$ and $\mathcal{V}$ an $(\mathcal{S}, \mathcal{V})$-pair if $\mathcal{S}$ is (C, A) invariant, $\mathcal{V}$ is (A, B) invariant, and $\mathcal{S} \subset \mathcal{V}$. In this paper, we use the notation $\Pi_{\mathcal{I}}$ and $|_{\mathcal{I}}$ to denote canonical projections and restrictions on a subspace $\mathcal{I}$, respectively.

With the preliminaries above, we recall the following necessary and sufficient conditions for solvability of DDPM.

THEOREM 3.1 (DDPM).
For the system Σ in (2.1)*, the following statements are equivalent:*

i. DDPM is solvable,
ii. $\mathcal{S}^* \subset \mathcal{V}^*$*,*
iii. there exists an $(\mathcal{S}, \mathcal{V})$ pair and matrices $F \in \mathcal{F}(\mathcal{V})$ and $L \in \mathcal{L}(\mathcal{S})$ such that $\operatorname{im}(G + LJ) \subset \mathcal{S} \subset \mathcal{V} \subset \ker(H + DF)$.

For systems with $D = 0$ and $J = 0$ the result is due to [8]. For systems with direct feedthrough matrices, the above result can be found in [7].

If DDPM is solvable, then there exists a dynamic controller of the form

$$\Sigma_C : \quad \begin{cases} \dot{w} = Pw + Qy, \\ u = Rw + Sy, \end{cases} \tag{3.1}$$

where $w(t) \in \mathbb{R}^w := \mathcal{W}$ that achieves disturbance decoupling. Within the class of all such controllers, the ones of minimal order have a state dimension

$$w = \dim(\mathcal{W}) = \min\{\dim(\mathcal{V}) - \dim(\mathcal{S}) \mid (\mathcal{S}, \mathcal{V}) \text{ satisfies iii. in Theorem 3.1}\}.$$

For details on the construction of the matrices in Σ_C based on the computation of $\mathcal{V}$ and $\mathcal{S}$, we refer to the papers [7, 8].

The synthesis of the controller is actually based on a simple observation when considering an extension of the original system Σ in (2.1). This extended system is given by:

$$\Sigma_e : \quad \begin{cases} \dot{x}_e = A_e x_e + B_e u_e + G_e d, \\ y_e = C_e x_e + J_e d, \\ \;z \;= H_e x_e + D_e u, \end{cases} \qquad u_e := \begin{bmatrix} u \\ u' \end{bmatrix}, \; x_e := \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathcal{X}_e, \; y_e := \begin{bmatrix} y \\ y' \end{bmatrix}, \tag{3.2}$$

where the matrices

$$A_e = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad B_e = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}, \quad G_e = \begin{bmatrix} G \\ 0 \end{bmatrix}, \quad C_e = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}, \quad J_e = \begin{bmatrix} J \\ 0 \end{bmatrix},$$

$H_e = \begin{bmatrix} H & 0 \end{bmatrix}$ and $D_e = \begin{bmatrix} D & 0 \end{bmatrix}$. Here, $\mathcal{X}_e := \mathcal{X} + \mathcal{W}$, so Σ_e is obtained from Σ in (2.1) by adding $w = \dim(\mathcal{W})$ integrators. The following lemma shows that the interconnecting a dynamic controller Σ_C with Σ is equivalent to interconnecting a static controller with the extended system Σ_e.

LEMMA 3.2. *The following two interconnections result in identical closed-loop transfer functions from disturbance signal d to output variable z:*
i. the dynamic controller (3.1) *interconnected with* Σ,
ii. the static controller $u_e = K_e y_e := \left[\begin{smallmatrix} S & R \\ Q & P \end{smallmatrix}\right] y_e$ *interconnected with* Σ_e.

The problem of finding a dynamic controller for Σ is therefore equivalent to finding a static controller for the extended system Σ_e. In either case, the controlled system is given by

$$\Sigma_{CL} : \quad \begin{cases} \dot{x}_e = (A_e + B_e K_e C_e) x_e + (G_e + K_e J_e) d, \\ \;z \;= (H_e + D_e K_e C_e) x_e + D_e K_e J_e d. \end{cases} \tag{3.3}$$

For the problem of DDPM, we are interested in the influence of the disturbance d to the output z for the interconnected, or closed-loop, system. With the representation Σ_{CL} of the controlled system in (3.3), this amounts to requiring the following conditions:

LEMMA 3.3. *The output z of the closed loop system Σ_{CL} in (3.3) will be independent of the disturbance d* if and only if *there exists a subspace $\mathcal{L}_e \subset \mathcal{X}_e$ such that:*

i. $\mathrm{im}(G_e + K_e J_e) \subset \mathcal{L}_e \subset \ker(H_e + D_e K_e C_e)$,

ii. $(A_e + B_e K_e C_e)\mathcal{L}_e \subset \mathcal{L}_e$,

iii. $D_e K_e J_e = 0$.

A subspace $\mathcal{L}_e \subset \mathcal{X}_e$ that satisfies the first *two* conditions of Lemma 3.3 for some matrix K_e is called (C_e, A_e, B_e) invariant [6]. This property is easily characterized according to the following lemma:

LEMMA 3.4. *A subspace $\mathcal{L}_e$ of $\mathcal{X}_e$ is (C_e, A_e, B_e) invariant if and only if $\mathcal{L}_e$ is (A_e, B_e) invariant* and *$\mathcal{L}_e$ is (C_e, A_e) invariant.*

Combining the previous two lemmas leads to the following observation.

THEOREM 3.5. *Consider the systems Σ in* (2.1) *and Σ_e in* (3.2).

i. If there exists an extension $\mathcal{X}_e$ of $\mathcal{X}$ and a subspace $\mathcal{L}_e \subset \mathcal{X}_e$ that is (C_e, A_e, B_e) invariant for Σ_e, then $(\mathcal{L}_e \cap \mathcal{X}, \Pi_{\mathcal{X}}\mathcal{L}_e)$ is an $(\mathcal{S}, \mathcal{V})$-pair for Σ.

ii. For any $(\mathcal{S}, \mathcal{V})$-pair there exists a vector space $\mathcal{Z}$ of $\dim(\mathcal{Z}) = \dim(\mathcal{V}) - \dim(\mathcal{S})$ *such that $\mathcal{L}_e := \mathcal{S} + \mathcal{Z}$ is (C_e, A_e, B_e) invariant and $(\mathcal{S}, \mathcal{V}) = (\mathcal{L}_e \cap \mathcal{X}, P_{\mathcal{X}}\mathcal{L}_e)$.*

The relation between (C_e, A_e, B_e) invariance for Σ_e and $(\mathcal{S}, \mathcal{V})$-pairs for Σ can now be used to give conditions on solvability of DDPM in Σ, as well as in Σ_e.

THEOREM 3.6. *Consider the systems Σ in* (2.1) *and Σ_e in* (3.2). *The following statements are equivalent:*

i. DDPM is solvable for Σ,

ii. there exists an $(\mathcal{S}, \mathcal{V})$-pair, a vector space $\mathcal{Z}$ of $\dim(\mathcal{Z}) = \dim(\mathcal{V}) - \dim(\mathcal{S})$ *and a matrix K_e such that $\mathcal{L}_e := \mathcal{S} + \mathcal{Z}$ satisfies the properties i., ii. and iii. in Lemma 3.3,*

iii. there exists a subspace $\mathcal{L}_e \subset \mathcal{X}_e$ such that $(\mathcal{L}_e \cap \mathcal{X}, P_{\mathcal{X}}\mathcal{L}_e)$ is an $(\mathcal{S}, \mathcal{V})$-pair that satisfies

$$\mathrm{im}(G + LJ) \subset \mathcal{S} \subset \mathcal{V} \subset \ker(H + DF)$$

for matrices $F \in \mathcal{F}(\mathcal{V})$ and $L \in \mathcal{L}(\mathcal{S})$.

4. Main results on approximation and DDPM.

The main result of this paper is a model reduction strategy that leaves solvability of DDPM invariant. Consider the following approximate system:

$$\hat{\Sigma}: \quad \begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u + \hat{G}d, \\ \hat{y} = \hat{C}\hat{x} + \hat{J}d, \\ \hat{z} = \hat{H}\hat{x} + \hat{D}u + \hat{E}d, \end{cases} \tag{4.1}$$

where $\hat{x}(t) \in \mathbb{R}^r := \hat{\mathcal{X}}$ with $r = \dim(\hat{\mathcal{X}}) < \dim(\mathcal{X}) = n$.

In view of the results in [5], we first consider the following two systems.

$$\hat{\Sigma}_{\mathcal{V}^*}: \quad \begin{cases} \dot{\hat{x}}' = \hat{A}'\hat{x}' + \hat{B}'u + \hat{G}'d, \\ y \ = \hat{C}'\hat{x}' + \hat{J}'d, \\ z \ = \hat{H}'\hat{x}' + \hat{D}'u, \end{cases} \qquad \text{and} \qquad \hat{\Sigma}_{\mathcal{S}^{*\perp}}: \quad \begin{cases} \dot{\hat{x}}'' = \hat{A}''\hat{x}'' + \hat{B}''u + \hat{G}''d, \\ y \ = \hat{C}''\hat{x}'' + \hat{J}''d, \\ z \ = \hat{H}''\hat{x}'' + \hat{D}''u, \end{cases}$$

where the system matrices are given by:

$$\begin{array}{llll}\hat{A}' := \Pi_{\mathcal{V}^*} A|_{\mathcal{V}^*}, & \hat{B}' := \Pi_{\mathcal{V}^*} B, & \hat{G}' := \Pi_{\mathcal{V}^*} G, & \hat{C}' := C|_{\mathcal{V}^*}, \\ \hat{J}' := J, & \hat{H}' := (H+DF)A|_{\mathcal{V}^*}, & \hat{D}' := (H+DF)B, & \text{and} \\ \hat{A}'' := \Pi_{\mathcal{S}^{*\perp}} A|_{\mathcal{S}^{*\perp}}, & \hat{B}'' := \Pi_{\mathcal{S}^{*\perp}} B, & \hat{G}'' := \Pi_{\mathcal{S}^{*\perp}} A(G+LJ), & \hat{C}'' := C|_{\mathcal{S}^{*\perp}}, \\ \hat{J}'' := C(G+LJ), & \hat{H}'' := H|_{\mathcal{S}^{*\perp}}, & \hat{D}'' := D, & \end{array}$$

with $\hat{x}'(t) \in \hat{\mathcal{X}}'$, $\hat{x}''(t) \in \hat{\mathcal{X}}''$, $F \in \mathcal{F}(\mathcal{V}^*)$ and $L \in \mathcal{L}(\mathcal{S}^*)$.

From these definitions it is immediate that $\hat{\Sigma}_{\mathcal{V}^*}$ and $\hat{\Sigma}_{\mathcal{S}^{*\perp}}$ have reduced orders $r' = \dim(\hat{\mathcal{X}}') = \dim(\mathcal{V}^*)$ and $r'' = \dim(\hat{\mathcal{X}}'') = \dim(\mathcal{S}^{*\perp})$, respectively. From these two reduced order systems, we immediately infer the following result concerning solvability of the problem of disturbance decoupling using partial measurements:

THEOREM 4.1. *The following two statements hold:*
i. DDPM is solvable for $\hat{\Sigma}_{\mathcal{V}^*}$,
ii. DDPM is solvable for $\hat{\Sigma}_{\mathcal{S}^{*\perp}}$.

Proof. Consider $\hat{\Sigma}_{\mathcal{V}^*}$. Set $\hat{F}' := F|_{\mathcal{V}^*}$ with $F \in \mathcal{F}(\mathcal{V}^*)$. Then, $\hat{H}' + \hat{D}'\hat{F}' = (H+DF)(A+BF)|_{\mathcal{V}^*} \subset (H+DF)\mathcal{V}^* = 0$, because $\mathcal{V}^*$ is (A,B) invariant and $\mathcal{V}^* \subset \ker(H+DF)$. Since the largest $(\hat{A}', \hat{B}')$ invariant subspace of $\hat{\mathcal{X}}'$ satisfies $\hat{\mathcal{V}}'^* \subset \ker(\hat{H}' + \hat{D}'\hat{F}')$, we obtain that $\hat{\mathcal{V}}^* = \hat{\mathcal{X}}'$. By definition, the smallest $(\hat{C}', \hat{A}')$ invariant subspace $\hat{\mathcal{S}}'^* \subset \hat{\mathcal{X}}'$, implying that DDPM is solvable for $\hat{\Sigma}_{\mathcal{V}^*}$.
For $\hat{\Sigma}_{\mathcal{S}^{*\perp}}$, set $\hat{L}'' := \Pi_{\mathcal{S}^{*\perp}} L$, with $L \in \mathcal{L}(\mathcal{S}^*)$. Then $\hat{G}'' + \hat{L}''\hat{J}'' = \Pi_{\mathcal{S}^{*\perp}}(A+LC)(G+LJ) \subset \Pi_{\mathcal{S}^{*\perp}}(A+LC)\mathcal{S}^* \subset \Pi_{\mathcal{S}^{*\perp}}\mathcal{S}^* = \{0\}$, hence $\operatorname{im}(\hat{G}''+\hat{L}''\hat{J}'') = 0$. So, the smallest $(\hat{C}'', \hat{A}'')$ invariant subspace $\hat{\mathcal{S}}''^* = \{0\}$, and therefore, by definition, $\hat{\mathcal{S}}''^* \subset \hat{\mathcal{V}}''^*$, which completes the proof. □

Let $F \in \mathcal{F}(\mathcal{V}^*)$ and $L \in \mathcal{L}(\mathcal{S}^*)$ and consider the subspaces $\mathcal{V}^*$ and $\mathcal{S}^*$. Define $\mathcal{L} := \mathcal{V}^* \cap \mathcal{S}^{*\perp}$ and consider the following reduced order system

$$\hat{\Sigma}_{\mathcal{L}} : \quad \begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u + \hat{G}d, \\ y = \hat{C}\hat{x} + \hat{J}d, \\ z = \hat{H}\hat{x} + \hat{D}u + \hat{E}d, \end{cases} \tag{4.2}$$

where $F \in \mathcal{F}(\mathcal{V}^*)$, $L \in \mathcal{L}(\mathcal{S}^*)$ and the state space matrices are given by:

$$\begin{array}{llll}\hat{A} := \Pi_{\mathcal{L}} A|_{\mathcal{L}}, & \hat{B} := \Pi_{\mathcal{L}} B, & \hat{G} := \Pi_{\mathcal{L}} A(G+LJ), & \hat{C} := C|_{\mathcal{L}}, \\ \hat{J} := C(G+LJ), & \hat{H} := (H+DF)A|_{\mathcal{L}}, & \hat{D} := (H+DF)B, & \end{array} \tag{4.3}$$

with $\hat{x}(t) \in \hat{\mathcal{X}}$ and $\dim(\hat{\mathcal{X}}) = \dim(\mathcal{L})$.

As is shown in the following result, the solvability of DDPM in $\hat{\Sigma}_{\mathcal{L}}$ only depends on the direct feed-through matrix $\hat{E}$:

THEOREM 4.2. *DDPM is solvable for* $\hat{\Sigma}_{\mathcal{L}}$ *if and only if* $\hat{E} = 0$.

Proof. The *only if* part of the claim is trivial. To prove the *if* part, suppose $\hat{E} = 0$ and define $\hat{F} = F|_{\mathcal{L}}$ and $\hat{L} = \Pi_{\mathcal{L}} L$. Introduce the controller (3.1) with $P = \hat{A} + \hat{L}\hat{C} + \hat{B}\hat{F}$, $Q = -\hat{L}$, $R = \hat{F}$ and $S = 0$. The interconnection of $\hat{\Sigma}_{\mathcal{L}}$ with this controller is described by the state space system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\hat{e}\\ \hat{x}\end{bmatrix} = \begin{bmatrix}\Pi_{\mathcal{L}}(A+LC)|_{\mathcal{L}} & 0\\ -\Pi_{\mathcal{L}}BF|_{\mathcal{L}} & \Pi_{\mathcal{L}}(A+BF)|_{\mathcal{L}}\end{bmatrix}\begin{bmatrix}\hat{e}\\ \hat{x}\end{bmatrix} + \begin{bmatrix}\Pi_{\mathcal{L}}(A+LC)(G+LJ)\\ \Pi_{\mathcal{L}}A(G+LJ)\end{bmatrix} d$$

$$z = \begin{bmatrix}-(H+DF)BF|_{\mathcal{L}} & (H+DF)(A+BF)|_{\mathcal{L}}\end{bmatrix}\begin{bmatrix}\hat{e}\\ \hat{x}\end{bmatrix} + \hat{E}d$$

where we introduced $\hat{e} = \hat{x} - w$ with w the controller state. Substituting the definitions of the matrices of the reduced order system and using the fact that $\mathcal{L} = \mathcal{S}^{*\perp} \cap \mathcal{V}^*$, we infer that $\Pi_{\mathcal{L}}(A+LC)(G+LJ) \subset \Pi_{\mathcal{L}}(A+LC)\mathcal{S}^* \subset \Pi_{\mathcal{L}}\mathcal{S}^* = 0$. It follows that the evolution of $\hat{e}$ will not depend on the disturbance d. In addition, since $(H+BF)(A+BF)|_{\mathcal{L}} \subset (H+DF)(A+BF)\mathcal{V}^* \subset (H+DF)\mathcal{V}^* = 0$ and $\hat{E} = 0$ it follows that the output $z = -(H+DF)BF|_{\mathcal{L}}\hat{e}$ which, when combined with the previous observation, shows that the output z will not depend on the disturbance d. This controller therefore achieves disturbance decoupling. □

The following theorem is the main result of the paper. It provides conditions under which the reduced order system $\hat{\Sigma}$ with state space matrices (4.3) has equivalent disturbance decoupling properties.

THEOREM 4.3. *Consider the reduced order system* $\hat{\Sigma}$ *in* (4.1) *with the state space matrices* (4.3) *and*

$$\hat{E} := \|\Pi_{\mathcal{S}^{*\perp}}(G+LJ)\| I,$$

where $F \in \mathcal{F}(\mathcal{V}^*)$, $L \in \mathcal{L}(\mathcal{S}^*)$, $\hat{x}(t) \in \hat{\mathcal{X}}$ *has* $\dim(\hat{\mathcal{X}}) = \dim(\mathcal{L})$, *and* $\|\cdot\|$ *denotes the matrix norm.*

i. If DDPM is solvable for Σ *in* (2.1), *then DDPM is solvable for* $\hat{\Sigma}$.
ii. If DDPM is solvable for $\hat{\Sigma}$ *in* (4.1) *and* $\exists F \in \mathcal{F}(\mathcal{V}^*)$ *such that* $F|_{\mathcal{S}^*} = 0$ *and* $\mathrm{im}(G) \subset \mathcal{V}^*$, *then DDPM is solvable for* Σ.

Proof. i. We know from Theorem 4.2 that DDPM is solvable for $\hat{\Sigma}_{\mathcal{L}}$ if and only if $\hat{E} = 0$. Since $L \in \mathcal{L}(\mathcal{S}^*)$ implies that $\mathrm{im}(G+LJ) \subset \mathcal{S}^*$ it follows that $\hat{E} = \|\Pi_{\mathcal{S}^{*\perp}}(G+LJ)\| I = 0$. Hence, DDPM is solvable for $\hat{\Sigma}$.
ii. Suppose the hypothesis holds. By Theorem 4.2, we have that $\hat{E} = 0$. Hence $\mathrm{im}(G+LJ) \subset \mathcal{S}^*$. Consider the controller (3.1) with $P = A + LC + BF$, $Q = -L$, $R = F$ and $S = 0$. Interconnection with Σ yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}e\\ x\end{bmatrix} = \begin{bmatrix}A+LC & 0\\ -BF & A+BF\end{bmatrix}\begin{bmatrix}e\\ x\end{bmatrix} + \begin{bmatrix}G+LJ\\ G\end{bmatrix} d, \qquad z = \begin{bmatrix}-DF & H+DF\end{bmatrix}\begin{bmatrix}e\\ x\end{bmatrix}.$$

Define $\mathcal{L}_e := \mathcal{S}^* \oplus \mathcal{V}^*$. Since under the given conditions and the fact that $\mathrm{im}(G + LJ) \subset \mathcal{S}^*$, one can see that $\mathrm{im}(\left[\begin{smallmatrix}G+LJ\\ G\end{smallmatrix}\right]) \subset \left[\begin{smallmatrix}\mathcal{S}^*\\ \mathcal{V}^*\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}A+LC & 0\\ -BF & A+BF\end{smallmatrix}\right]\left[\begin{smallmatrix}\mathcal{S}^*\\ \mathcal{V}^*\end{smallmatrix}\right] \subset \left[\begin{smallmatrix}\mathcal{S}^*\\ \mathcal{V}^*\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}-DF & H+DF\end{smallmatrix}\right]\left[\begin{smallmatrix}\mathcal{S}^*\\ \mathcal{V}^*\end{smallmatrix}\right] = 0$. By Lemma 3.3 we have that Σ is DDPM solvable. □

5. Conclusions and future research.

In this paper, we have addressed the problem of model reduction for control purposes. We focused on a reduction strategy that keeps the solvability of the problem of disturbance decoupling with partial measurement feedback (DDPM) invariant. That means that after reduction the problem of DDPM is still solvable, and that the resulting controller (obtained using the reduced order system) solves DDPM after interconnection with the original system. The main problem of this paper is to find conditions whether the original system is DDPM solvable if and only if the reduced order system is DDPM solvable.

For future research, model reduction strategies that keep other control strategies invariant need to be investigated, as e.g. almost disturbance decoupling problems (ADDP).

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