A numerical condition for the physical realizability of a quantum linear system *

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1 Introduction

Quantum feedback control is an active research area with applications in quantum optics, quantum communications, quantum computing and precision measurement [1–9]. Of particular interest is coherent quantum feedback control whereby a controller is implemented as a fully quantum system coupled to a quantum plant. The advantage of this arrangement is that it avoids the destruction of quantum information which necessarily occurs during the measurement process in the case of a classical controller [1, 2, 5, 6].

Key to the problem of coherent quantum feedback control is the question of whether a particular controller transfer function is able to be implemented as a quantum system. This problem is referred to as *physical realizability* and is addressed in [1, 10].

When designing LTI controllers, it is normal for the transfer function of the controller to determine the closed loop performance of the system rather than the particular state space realization of the controller. As such, the question of whether a particular transfer function is physically realizable is, in general, of greater interest than whether a particular state space implementation is physically realizable.

This papers further addresses the problem of whether a particular transfer function is physically realizable with a minimal number of number of additional

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quantum noises. The reader might consider the question of physical realizability without any additional quantum noises to be of greater interest, however, it turns out that systems described by strictly proper transfer functions are never physically realizable without additional noises.

It has already been shown in [1], that an arbitrary LTI system is always physically realizable if additional quantum noise components are permitted in the implementation. Adding unnecessary noise is undesirable, and the paper [9] addresses the problem of whether it is possible to implement a controller with a specified strictly proper transfer function as a quantum system with only the minimum number of additional quantum noises. One of the results given in [9] is a condition for physical realizability with a minimal number of additional noises. This condition is given in terms of a certain non-standard Riccati equation. In this paper we provide conditions for the existence of a suitable solution to this Riccati equation. This leads to a numerical solution to the physical realizability problem under consideration.

2 Quantum System Model

As in [1], the linear quantum systems under consideration are assumed to be noncommutative stochastic systems described by quantum stochastic differential equations (QSDEs) of the form

$$\begin{aligned} \mathrm{d}x(t) &= Ax(t)\,\mathrm{d}t + \begin{bmatrix} B_1 & B \end{bmatrix} \begin{bmatrix} \mathrm{d}v(t) \\ \mathrm{d}u(t) \end{bmatrix}; \qquad x(0) = x_0 \\ \mathrm{d}y(t) &= Cx(t)\,\mathrm{d}t + \begin{bmatrix} D_1 & 0_{n_y \times n_u} \end{bmatrix} \begin{bmatrix} \mathrm{d}v(t) \\ \mathrm{d}u(t) \end{bmatrix} \end{aligned} \tag{1}$$

where $x(t) = \begin{bmatrix} x_1(t) & \cdots & x_n(t) \end{bmatrix}^T$ is a column vector of n self-adjoint system variables. The noise $v(t) = \begin{bmatrix} v_1(t) & \cdots & v_{n_v}(t) \end{bmatrix}^T$ is a vector of noncommutative Wiener processes (in vacuum states) with Ito products $\mathrm{d}v(t)\,\mathrm{d}v^T(t) = F_v\,\mathrm{d}t$ where F_v is non-negative Hermitian. $\mathrm{d}u(t)$ is a column vector of signals of the form $\mathrm{d}u(t) = \beta_u(t)\,\mathrm{d}t + \mathrm{d}\tilde{u}(t)$ where $\tilde{u}(t)$ is the noise part of $u(t)$ (with Ito products $\mathrm{d}\tilde{u}(t)\,\mathrm{d}\tilde{u}^T(t) = F_{\tilde{u}}\,\mathrm{d}t$ where $F_{\tilde{u}}$ is non-negative Hermitian) and $\beta_u(t)$ is the adapted, self adjoint part of $u(t)$. $u(t)$ represents the input to the system. n, n_v, n_u and n_y are all assumed to be even, and A, B, C, B_1 and D_1 are appropriately dimensioned real matrices describing the dynamics of the system. For further details see [1].

3 Physical Realizability

3.1 Definition

As in [1,2,9,10], the concept of *physically realizable* means that the system dynamics described by (1) correspond to those of an open quantum harmonic oscillator.

Definition 1. *By the* canonical commutation relations *we mean that the system variables x satisfy the commutation relations $\left[x_i(t), x_j(t)\right] = 2i\Theta_{ij}$ where Θ is a block diagonal matrix with each diagonal block equal to $J = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$.*

Definition 2. *By* degenerate canonical commutation relations*, we mean that for the system (1), Θ as defined above is a block diagonal matrix with a zero matrix block followed by diagonal block's equal to J as above, along the diagonal.*

The canonical commutation relations describe systems of a purely quantum nature whereas the degenerate canonical commutation relations describe hybrid systems consisting of both classical and quantum components.

Definition 3. *(See [1, Definition 3.1]). The system described by (1) is an* open quantum harmonic oscillator *if* Θ *is canonical and there exist a quadratic Hamiltonian* $H = \frac{1}{2}x(0)^T Rx(0)$*, with a real, symmetric,* $n \times n$ *matrix* R*, and a coupling operator* $L = \Lambda x(0)$*, with complex-valued* $\frac{1}{2}(n_v + n_u) \times n$ *coupling matrix* Λ*, such that the matrices* A*,* $\begin{bmatrix} B_1 & B \end{bmatrix}$*,* C *and* $\begin{bmatrix} D_1 & 0_{n_y \times n_u} \end{bmatrix}$ *are given by:*

$$A = 2\Theta\left(R + \mathfrak{Im}\left(\Lambda^\dagger \Lambda\right)\right); \tag{2}$$

$$\begin{bmatrix} B_1 & B \end{bmatrix} = 2i\Theta \begin{bmatrix} -\Lambda^\dagger & \Lambda^T \end{bmatrix} \Gamma; \tag{3}$$

$$C = P^T \begin{bmatrix} \Sigma_{n_y} & 0 \\ 0 & \Sigma_{n_y} \end{bmatrix} \begin{bmatrix} \Lambda + \Lambda^\# \\ -i\Lambda + i\Lambda^\# \end{bmatrix}; \tag{4}$$

$$\begin{bmatrix} D_1 & 0_{n_y \times n_u} \end{bmatrix} = \begin{bmatrix} I_{n_y \times n_y} & 0_{n_y \times (n_v + n_u - n_y)} \end{bmatrix}. \tag{5}$$

Here: $\Gamma_{n_v \times n_v} = P \operatorname{diag}(M)$*;* $M = \frac{1}{2}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$*;* $\Sigma_{n_y} = \begin{bmatrix} I_{\frac{1}{2}n_y \times \frac{1}{2}n_y} & 0_{\frac{1}{2}n_y \times \frac{1}{2}(n_v + n_u - n_y)} \end{bmatrix}$*;* P *is the appropriately dimensioned square permutation matrix such that* $P \begin{bmatrix} a_1 & a_2 & \cdots & a_{2m} \end{bmatrix} = \begin{bmatrix} a_1 & a_3 & \cdots & a_{2m-1} a_2 & a_4 & \cdots & a_{2m} \end{bmatrix}$ *and* $\operatorname{diag}(M)$ *is the appropriately dimensioned square block diagonal matrix with the matrix* M *occurring along the diagonal. (Note: dimensions of* P *and* $\operatorname{diag}(M)$ *can always be determined from the context in which they appear.)* $\mathfrak{Im}(.)$ *denotes the imaginary part of a matrix and* † *denotes the complex conjugate transpose of a matrix.*

Definition 4. *(See [1, Definition 3.3]). The system (1) is said to be physically realizable if one of the following holds:*

1. Θ *is canonical and (1) represents an open quantum harmonic oscillator.*

2. Θ *is degenerate canonical and there exists an augmentation which, after a suitable relabeling of the components, represents the dynamics of an open quantum harmonic oscillator [1].*

In this paper we consider the problem of implementing a linear quantum system with a given transfer function as a fully quantum system and as such focus on the case where Θ is canonical.

3.2 Previous Results

In [1], it was demonstrated that by incorporating additional quantum noises, an arbitrary linear time invariant system could always be physically realized as a quantum system. In particular, the following lemma relating to the physical realizability of a purely quantum controller with canonical commutation relations was proven.

Lemma 5. *(See [1, Lemma 5.6]). Suppose* $F_u = I + iJ$*,* $F_v = I + iJ$*,* A, B *and* C *are such that* $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times n_u}, C \in \mathbb{R}^{n_y \times n}$*, and* $\Theta = \operatorname{diag}(J)$ *is canonical. Then there exists an even integer* $n_v \geq n_y$ *and matrix* $B_1 \in \mathbb{R}^{n \times n_v}$ *such that the system (1) is physically realizable.*

Remark 1. *By* minimal additional noises *we mean that* $n_v = n_y$. *It follows from [1, Theorem 3.4] that the number of outputs* n_y *is a lower bound on the number of additional noises* n_v*, necessary for physical realizability for a system described by a strictly proper transfer function.*

In general, the incorporation of additional noises is undesirable, and in [9] the question of how many additional noises are required to implement a strictly proper LTI Quantum System was addressed. Drawing on the results contained in [1], in [9] an upper bound on the number of additional noises required was given. Further to this, a condition in terms of a certain non-standard Riccati equation was obtained for when a given transfer function (rather than a specific state space realization) could be physically realized using only a minimum number of additional quantum noises. The main results of [9] are reproduced here for the sake of completeness.

Theorem 6. *(See [9, Theorem 1]). Consider an LTI system of the form (1) where* A, B *and* C *are given and the system commutation matrix* Θ *is canonical. There exists* B_1 *and* D_1 *such that the system is physically realizable with the number of quantum noises in* $\mathrm{d}v$ *equal to* $n_u + 2(n - n_\lambda)$ *where* n_λ *is the multiplicity of the least (i.e. most negative) eigenvalue of the matrix* $i\left(\Theta B \Theta B^T \Theta - \Theta A - A^T \Theta - C^T \Theta C\right)$.

In the special case that $n_\lambda = n$, it follows that $n_v = n_u$, which is the minimum number of quantum noises which need to be added for physically realizability.

The second result of [9] relates to implementing an LTI quantum system with a specified strictly proper transfer function. This is equivalent to allowing a state transformation on the state space description of the system.

Theorem 7. *(See [9, Theorem 2]). Consider an LTI system of the form (1) where* A, B *and* C *are given and the system commutation matrix* Θ *is canonical. Suppose the Riccati equation*

$$\Psi B \Theta B^T \Psi - A^T \Psi - \Psi A - C^T \Theta C = 0 \tag{6}$$

has a non-singular solution Ψ *which is skew symmetric and suppose that there exists a real non-singular matrix* T *such that* $\Psi = T^T \Theta T$. *Then there exists a system described by* $\left\{\tilde{A}, \tilde{B}, \tilde{C}\right\}$ *with the same transfer function as the system* $\{A, B, C\}$ *which can be physically realized with the minimum number of additional noises* $n_v = n_u$*, where* $\tilde{A} = TAT^{-1}$*;* $\tilde{B} = TB$*;* $\tilde{C} = CT^{-1}$.

4 Main Result

In this paper, we obtain conditions for when the Riccati equation (6) has a non-singular skew symmetric solution Ψ, of the form $\Psi = T^T \Theta T$ (T non-singular). The proof given below for the existence of a solution to the Riccati equation (6) closely follows that for the existence of a stabilizing solution to a standard ARE as given, for example, in [11, Theorem 13.5]. This result also leads to a numerical procedure for solving the ARE (6) and testing for physical realizability with minimal additional noises.

For convenience, define $R = -B \Theta B^T$, $Q = C^T \Theta C$, and $X = \Psi$. From (6):

$$A^T X + XA + XRX + Q = 0. \tag{7}$$

Note that Q and R are skew symmetric and (6) and (7) are equivalent.

Define $H = \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix}$, and $P = -i\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. Note that $P^{-1} = P^{\dagger}$, (PH) is skew symmetric, and $P^{-1}HP = P^{\dagger}HP = -H^{\dagger}$. That is, H and $-H^{\dagger}$ are similar, from which it follows that λ is an eigenvalue of H if and only if $-\lambda$ is, i.e. the eigenvalues of H are symmetric about the imaginary axis.

Assume H has no eigenvalues on the imaginary axis. Let $\chi_-(H)$ be the n-dimensional spectral subspace of H corresponding to its negative eigenvalues. By finding a basis for $\chi_-(H)$, stacking the basis vectors to form a matrix, and partitioning the matrix we can write $\chi_-(H) = \text{Im}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$; $X_1, X_2 \in \mathbb{C}^{n\times n}$.

Assume X_1 is non-singular or equivalently that $\chi_-(H)$ and $\text{Im}\begin{bmatrix} 0 \\ I \end{bmatrix}$ are complementary and define $X = X_2X_1^{-1}$. X is uniquely determined by H; we shall denote this function *Ric*, i.e. $X = \mathit{Ric}(H)$ with the domain *dom(Ric)* consisting of Hamiltonian matrices H satisfying two properties: that H has no purely imaginary eigenvalues, and that X_1 is non-singular.

Theorem 8. *Suppose* $H \in \text{dom(Ric)}$ *and* $X = \text{Ric}(H)$. *Then* X *is skew-symmetric and solves the algebraic Riccati equation* $A^TX + XA + XRX + Q = 0$.

Proof. Let X_1,X_2 be as above. There exists a stable matrix $H_- \in \mathbb{C}^{n\times n}$ such that

$$H\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} H_-. \tag{8}$$

Pre-multiply (8) by $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}^T P$ to obtain $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}^T PH\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}^T P\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} H_-$. Since PH is skew-symmetric, so are both sides of the above equation. From the right-hand side: $(X_2^TX_1 + X_1^TX_2)H_- = -H_-^{\dagger}(X_2^TX_1 + X_1^TX_2)^T$. This is a Lyapanov equation. Since H_- is stable the unique solution is $X_2^TX_1 + X_1^TX_2 = 0$. That is, $X_1^TX_2$ is skew symmetric, and since X_1 is non-singular, $X = (X_1^{-1})^T(X_1^TX_2)X_1^{-1}$ is also skew-symmetric.

It remains to be shown that X is a solution to (7). Post-multiplying (8) by X_1^{-1} we obtain $H\begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} X_1H_-X_1^{-1}$ and pre-multiplying by $\begin{bmatrix} X & -I \end{bmatrix}$ gives $\begin{bmatrix} X & -I \end{bmatrix} H\begin{bmatrix} I \\ X \end{bmatrix} = 0$, which is precisely (7). □

Remark 2. *This proof also leads to a numerical procedure for solving the ARE (6) and hence solving the physical realizability problem under consideration by solving the eigenvalue, eigenvector problem for the matrix H. Then, the following corollary is the main result.*

Corollary 9. *Consider an LTI system of the form (1) with* A, B *and* C *given and with the canonical commutation matrix* Θ *and suppose* $H \in \text{dom(Ric)}$ *and* $X = \text{Ric}(H)$ *is non-singular. There exists a system* $\left\{\tilde{A}, \tilde{B}, \tilde{C}\right\}$ *with the same transfer function as the system* $\{A, B, C\}$ *which is physically realizable with the minimum additional noises* $n_v = n_u$, *where* $\tilde{A} = TAT^{-1}$, $\tilde{B} = TB$, *and* $\tilde{C} = CT^{-1}$.

Proof. By applying *Theorem 3*, observe that (7) and equivalently (6) have non-singular skew symmetric solutions, $\Psi = X$. Also, for any non-singular $2m \times 2m$ real skew-symmetric matrix there exists a non-singular real matrix P for which $X = P^T\Theta P$ [12, Corollary 8.24]. The result follows by applying *Theorem 2.* □

Remark 3. *The following provides a numerical algorithm to construct* P *such that* $X = P^T \Theta P$.

Since X *is skew-symmetric,* $X = V \Delta V^\dagger$, *where* V *is unitary with columns precisely the eigenvectors of* X *and* Λ *is diagonal with diagonal elements precisely the eigenvalues of* X *which are purely imaginary and occur in complex conjugate pairs. For every* v *an eigenvector corresponding to* λ, *its complex conjugate* $\bar{v}$ *is also an eigenvector and corresponds to* $\bar{\lambda}$. *If necessary, reorder the columns of* V *and corresponding entries of* Λ *such that these complex conjugate pairs are adjacent:* $V = \left[v_1 \; \bar{v}_1 \; \dots \; v_{2n-1} \; \bar{v}_{2n-1} \right]$

Similarly $\Theta = \tilde{V} \tilde{\Lambda} \tilde{V}^\dagger$ *where* V *is a block diagonal matrix with repeating blocks* $\frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$ *and* Λ *is a diagonal matrix with alternating entries* i *and* $-i$. *Observe that there exists a diagonal matrix* T *such that* $\Lambda = T \tilde{\Lambda} T$ *and the elements of* T, t_n *are positive real and satisfy* $t_{2n-1} = t_{2n}$. *We now have* $X = VT\tilde{V}^\dagger \Theta \tilde{V} T V^\dagger$. *Define* $P = \tilde{V} T V^\dagger$, *then* $X = P^\dagger \Theta P$

Observe that $\left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix} t & 0 \\ 0 & t \end{smallmatrix}\right]$ *commute, therefore* $\tilde{V}$ *and* T *commute. We claim that* $P = \tilde{V} V^\dagger T$ *is real and demonstrate this by considering* $(\tilde{V} V^\dagger)^\dagger = V \tilde{V}^\dagger$ $= \left[v_1 \; \bar{v}_1 \; v_3 \; \bar{v}_3 \; \dots \right] diag\left(\frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right] \right) = \frac{1}{\sqrt{2}} \left[(v_1+\bar{v}_1) \; -i(v_1-\bar{v}_1) \; (v_3+\bar{v}_3) \; -i(v_3-\bar{v}_3) \dots \right]$ *which is real. Therefore* P *as constructed above is real and* $X = P^T \Theta P$.

5 Example

In this section we consider a system from [1, Section VII,D], in which an example of classical-quantum controller synthesis is given, where the controller is implemented as a degenerate canonical controller with both classical and quantum degrees of freedom. We show that the particular state space representation of the system given cannot be physically realized as a fully quantum system with the minimum number of additional noises, however there exists another state space realization of the same transfer function which can. Our example demonstrates the numerical method for solving the Riccati equation, and obtaining the system to be implemented.

Consider a system of the form (1) with $A = \begin{bmatrix} -1.3894\, I_{2\times2} & -0.4472\, I_{2\times2} \\ -0.2\, I_{2\times2} & -0.25\, I_{2\times2} \end{bmatrix}$; $B = \begin{bmatrix} -0.4472\, I_{2\times2} \\ 0_{2\times2} \end{bmatrix}$; $C = \left[-0.4472\, I_{2\times2} \; 0_{2\times2} \right]$; $n = 4$; $n_u = 2$ and $n_y = 2$.

Applying *Theorem 6*: $S = i\left(\Theta B \Theta B^T \Theta - \Theta A - A^T \Theta - C^T \Theta C\right)$ $= \begin{bmatrix} 0 & 2.3788i & 0 & 0.6472i \\ -2.3788i & 0 & -0.6472i & 0 \\ 0 & 0.6472i & 0 & 0.5i \\ -0.6472i & 0 & -0.5i & 0 \end{bmatrix}$ with least eigenvalue -2.5802 which has multiplicity 1. Therefore, physically realizing this system requires $n_v = n_u + 2(n - n_\lambda) = 8$ quantum noises, which is strictly greater than the minimum number of additional quantum noises $n_v = n_u$.

We now apply our main result to show that it is possible to realize the corresponding transfer function of this system as a quantum system with the minimum number of additional quantum noises. In this case, we calculate

$$R = \begin{bmatrix} 0 & -0.2 & 0 & 0 \\ 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \qquad Q = \begin{bmatrix} 0 & 0.2 & 0 & 0 \\ -0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

$$H = \begin{bmatrix} -1.389 & 0 & -0.447 & 0 & 0 & -0.2 & 0 & 0 \\ 0 & -1.389 & 0 & -0.447 & 0.2 & 0 & 0 & 0 \\ -0.2 & 0 & -0.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.2 & 0 & -0.25 & 0 & 0 & 0 & 0 \\ 0 & -0.2 & 0 & 0 & 1.389 & 0 & 0.2 & 0 \\ 0.2 & 0 & 0 & 0 & 0 & 1.389 & 0 & 0.2 \\ 0 & 0 & 0 & 0 & 0.447 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.447 & 0 & 0.25 \end{bmatrix}.$$

The eigenvalues of H are: -1.4496, -1.4496, -0.1745, -0.1745, 0.1745, 0.1745, 1.4496 and 1.4496.

The matrix $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ is formed by stacking the eigenvectors corresponding to the four eigenvalues with negative real parts:

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} -0.3520 & -0.9838 & 0.0223 & -0.0409 \\ 0 & 0 & 0.9835 & 0.3496 \\ 0.93290 & -0.1640 & 0.0037 & 0.1083 \\ 0 & 0 & 0.1640 & -0.9266 \\ 0 & 0 & 0.0706 & 0.0517 \\ 0.0520 & 0.0706 & -0.0016 & 0.0060 \\ 0 & 0 & -0.0186 & -0.0544 \\ -0.0548 & -0.0186 & 0.0004 & -0.0064 \end{bmatrix}$$

In this example, X_1 is non-singular, therefore applying *Theorem 3*,

$$X = X_2 X_1^{-1} = \begin{bmatrix} 0 & 0.0763 & 0 & -0.0270 \\ -0.0763 & 0 & 0.0270 & 0 \\ 0 & -0.0270 & 0 & 0.0486 \\ 0.0270 & 0 & -0.0486 & 0 \end{bmatrix}$$

is a skew-symmetric solution to (6) and (7), as can be confirmed by substitution. Finally, we note that X is non-singular and apply *Corollary 1*: $X = T^T \Theta T$ where $T = \begin{bmatrix} 0 & 0.2599 & 0 & -0.1587 \\ -0.2599 & 0 & 0.1587 & 0 \\ -0.0933 & 0 & -0.1529 & 0 \\ 0 & -0.9033 & 0 & -0.1529 \end{bmatrix}$: $\tilde{A} = TAT^{-1} = \begin{bmatrix} -0.7922 & 0 & 0 & 1.3232 \\ 0 & -0.7922 & -1.3232 & 0 \\ 0 & -0.3132 & -0.8472 & 0 \\ 0.3123 & 0 & 0 & -0.8472 \end{bmatrix}$; $\tilde{B} = TB = \begin{bmatrix} 0 & -0.1162 \\ 0.1162 & 0 \\ 0.0417 & 0 \\ 0 & 0.0417 \end{bmatrix}$; $\tilde{C} = CT^{-1} = \begin{bmatrix} 0 & 1.2533 & 1.3010 & 0 \\ -1.2533 & 0 & 0 & 1.3010 \end{bmatrix}$. Here, the matrix T has been constructed using the method described in *Remark 2*.

The system described by $\left\{ \tilde{A}, \tilde{B}, \tilde{C} \right\}$ implements the same transfer function as the system $\{A, B, C\}$ but unlike the later, it can be physically realized with the minimum additional noises $n_v = n_u = 2$.

6 Conclusion

An arbitrary linear time invariant system can always be implemented as a quantum system if additional quantum noises are allowed in the implementation [1]. However, incorporating unnecessary noises is undesirable and in [9] an expression for an upper bound on the number of required additional noises was given. This paper gives a numerical method for determining if a given transfer function matrix can be implemented as a linear quantum system with minimal additional noises.

This result has direct applications in coherent quantum feedback control where we wish to know whether it is possible to implement a given controller that is the outcome of a design process using the minimum number of additional quantum noises.

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