GEOMETRIC DECOMPOSITION AND POTENTIAL-BASED REPRESENTATION OF NONLINEAR SYSTEMS

M. GUAY*, N. HUDON†, AND K. HÖFFNER‡

Abstract. This paper considers the problem of representing a sufficiently smooth nonlinear dynamical as a potential-driven system. The problem is related to recent contributions where feedback controllers are designed for gradient systems, generalized Hamiltonian systems and systems given in Brayton–Moser form.The approach proposed in the present note is based on a decomposition of a differential one-form associated to the given vector field into its exact and anti-exact components, and into its coexact and anti-coexact components. In particular, a dual operator to the standard homotopy operator, which inverts locally the exterior differential operator, is defined and used in the present paper to invert the codifferential operator. A study of Brayton–Moser representation is presented to illustrate the proposed approach.

Key words. Nonlinear systems, systems representation, (dual) homotopy decomposition.

AMS subject classifications. 93A30, 93B27

1. Introduction. Analysis and control design using on physically-based nonlinear representations, such as gradient systems [2], generalized Hamiltonian systems [1], and systems given by Brayton–Moser equations [5], are now central to nonlinear control theory and practice. For applications where the concept of free energy is ill-defined, such representations are not available *a priori* and the problem of deriving a potential-based representation proved difficult to be solved. In the present note, we consider a nonlinear dynamical system of the form

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $f \in \mathcal{C}^k$, with $k \geq 2$ and we assume that the origin is an isolated equilibrium for $f(x)$. The general problem considered in the present paper is to represent the system (1.1) as

$$\dot{x} = -Q(x)\frac{\partial P^T(x)}{\partial x}, \tag{1.2}$$

where $P(x)$ is a potential for the system and $Q(x)$ is a structure matrix.

The approach proposed in the present paper seeks to refine the technique proposed originally in [6], which was based on the application of a homotopy operator (see Section 3) on a differential one-form associated to the system (1.1), to find a potential for a given system using the Poincaré lemma [10, 5, 4]. In particular, we characterize further the structure of the dynamics by proposing a dual to the homotopy operator (Section 4). This operator locally inverts the codifferential operator [8]. In particular, the proposed approach can be related to the representation of smooth nonlinear dynamics as the sum of a gradient system and $(n-1)$ Hamiltonian systems, as presented for example in [9] using Hodge theory.

*Corresponding author, martin.guay@chee.queensu.ca. Department of Chemical Engineering, Queen's University, Kingston, ON, Canada, K7L 3N6.

†School of Chemical Engineering, The University of New South Wales, UNSW Sydney, NSW 2052, Australia.

‡Process Systems Engineering Laboratory, Department of Chemical Engineering, Massachusetts Institute of Technology, Cambridge, MA, U.S.A. 02139.

The paper is divided as follows. A review of exterior calculus and differential forms is presented in Section 2. The homotopy operator is reviewed in Section 3 and a dual homotopy operator, which allows us to invert the codifferential operator is proposed in Section 4. An application of the proposed construction to the problem of Brayton–Moser representation is given in Section 5. Conclusions and areas for further research are given in Section 6.

2. Preliminaries. Geometric elements expressed in the formalism of exterior calculus and differential forms on $\mathbb{R}^n$ are given in the present section following [7]. We denote a smooth vector field in $\Gamma(\mathbb{R}^n)$ as $X(x) = \sum_{i=1}^n v^i(x)\frac{\partial}{\partial_{x_i}}$ and a smooth differential one-form in $\Lambda^1(\mathbb{R}^n)$ as $\omega(x) = \sum_{i=1}^n \omega_i(x)dx_i$, where $v^i(x)$ and $\omega_i(x)$ are smooth functions on $\mathbb{R}^n$. The standard basis for vectors in $\Gamma(\mathbb{R}^n)$ and one-forms in $\Lambda^1(\mathbb{R}^n)$ are denoted by $\frac{\partial}{\partial x_i}$ and dx_i, respectively. The wedge product $\wedge$ is defined on $\Lambda^1(\mathbb{R}^n) \times \Lambda^1(\mathbb{R}^n)$ by the requirements

$$dx_i \wedge dx_j = -dx_j \wedge dx_i$$
$$dx_i \wedge f(x)dx_j = f(x)dx_i \wedge dx_j$$

for all smooth functions $f(x)$ and by

$$\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma,$$

for all $\alpha, \beta, \gamma \in \Lambda(\mathbb{R}^n) = \bigoplus_{k=0}^n \Lambda^k(\mathbb{R}^n)$. If $\alpha \in \Lambda^k(\mathbb{R}^n)$, then we write $\deg \alpha = k$. Note that in this context, the space of 0-forms, $\Lambda^0(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$ is the space of smooth functions on $\mathbb{R}^n$. The interior product $\lrcorner$ is a map

$$\lrcorner : \Gamma^\infty(\mathbb{R}^n) \times \Lambda^k(\mathbb{R}^n) \to \Lambda^{k-1}(\mathbb{R}^n), \quad 0 \le k \le n,$$

with the following properties for all $\alpha, \beta \in \Lambda^k(\mathbb{R}^n)$, $X, X_1, X_2 \in \Gamma(\mathbb{R}^n)$ and $f, g \in \Lambda^0(\mathbb{R}^n)$:

1. $X \lrcorner f = 0$;
2. $X \lrcorner \omega = \omega(X), \forall \omega \in \Lambda^1(\mathbb{R}^n)$;
3. $X \lrcorner (\alpha + \beta) = X \lrcorner \alpha + X \lrcorner \beta$;
4. $(fX_1 + gX_2) \lrcorner \alpha = f \cdot (X_1 \lrcorner \alpha) + g \cdot (X_2 \lrcorner \alpha)$;
5. $X \lrcorner (\alpha \wedge \beta) = (X \lrcorner \alpha) \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge (X \lrcorner \beta)$;
6. $X_1 \lrcorner (X_2 \lrcorner \alpha) = -X_2 \lrcorner (X_1 \lrcorner \alpha)$.

The last property leads to the following composition rule for interior product: $X \lrcorner (X \lrcorner \alpha) = 0$. The differential operator d is the unique operator on $\Lambda(\mathbb{R}^n)$,

$$d : \Lambda^k(\mathbb{R}^n) \to \Lambda^{k+1}(\mathbb{R}^n), \quad 0 \le k \le n-1,$$

with the following properties:

1. $d(\alpha + \beta) = d\alpha + d\beta$;
2. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$;
3. $df = \sum_i (\frac{\partial f}{\partial x_i}) dx_i, \ \forall f(x) \in \Lambda^0(\mathbb{R}^n)$;
4. $d \circ d\alpha = 0$.

If a one-form ω is closed, *i.e.*, if $d\omega = 0$, it can be shown that it is also locally exact, by virtue of the Poincaré Lemma, and the system is conservative (in particular, the dynamics is generated by the gradient of a potential function). However, if the one-form is not closed, ω can be decomposed as the sum of an exact component and an anti-exact component, using a locally-defined homotopy operator presented in the next section.

The derivation of a differential one-form associated to the drift vector field $f(x)$ relies on the canonical Riemannian metric in $\mathbb{R}^n$, given as $g = dx_1 \otimes dx_1 + \ldots + dx_n \otimes dx_n$, with its associated volume form in $\Lambda^n(\mathbb{R}^n)$, expressed as $\mu = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$. The central element to be exploited in the sequel is the divergence of the vector field $f(x)$, computed as follows [7]. A $(n-1)$ differential form j is first obtained by taking the interior product of the volume μ with respect to the drift vector field $f(x)$, *i.e.*,

$$j = \left(\sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i} \right) \lrcorner \mu = \sum_{i=1}^{n} (-1)^i f_i(x) dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n, \tag{2.1}$$

where $\widehat{dx_i}$ denotes a removed element such that j is a $(n-1)$ form. Taking the exterior derivative of j, and by the property of the wedge product that $dx_i \wedge dx_i = 0$, we obtain,

$$dj = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x) dx_1 \wedge \ldots \wedge dx_n = \text{div} f(x) \mu. \tag{2.2}$$

A differential one-form $\omega \in \Lambda^1(\mathbb{R}^n)$ that encodes the divergence of the drift vector field $f(x)$ is obtained by using the Hodge star operator $\star$ of the $(n-1)$ form j, *i.e.*,

$$\omega = \star j = \star(f(x) \lrcorner \mu) = (-1)^{n-1} \sum_{i=1}^{n} f_i(x) dx_i. \tag{2.3}$$

The Hodge star operator $\star : \Lambda^k(\mathbb{R}^n) \to \Lambda^{n-k}(\mathbb{R}^n)$ has the following properties [8, Chapter 4]. For any f and g in $C^\infty(\mathbb{R}^n)$ and for any α and β in $\Lambda^k(\mathbb{R}^n)$,

1. $\star(f\alpha + g\beta) = f \star \alpha + g \star \beta$;
2. $\star \star \alpha = (-1)^{k(n-k)} \alpha$;
3. $\alpha \wedge \star\beta = \beta \wedge \star\alpha = \langle \alpha, \beta \rangle \mu$;
4. $\star(\alpha \wedge \star\beta) = \star(\beta \wedge \star\alpha) = \langle \alpha, \beta \rangle$;
5. $\langle \star\alpha, \star\beta \rangle = \langle \alpha, \beta \rangle$.

Following [8], we also define the codiffential operator, $\delta : \Lambda^k(\mathbb{R}^n) \to \Lambda^k(\mathbb{R}^{n-1})$, as

$$\delta = (-1)^{n(k+1)+1} \star d \star, \tag{2.4}$$

which has the following properties:

1. $\star\delta = (-1)^k d\star$.
2. $\delta\star = (-1)^{k+1} \star d$.
3. $\delta \circ \delta = 0$.

A differential form $\alpha \in \Lambda^k(\mathbb{R}^n)$ is said to be coclosed if $\delta\alpha = 0$. It is said to be coexact if there exists a $k+1$ form β such that $\delta\beta = \alpha$. The dual homotopy operator proposed in Section 4 seeks to invert the codifferential operator to compute the coclosed part of a differential form in the same way that the homotopy operator is used to computed the exact part of a differential form by inverting the exterior derivative. Relative to an inner product $(\cdot, \cdot)$ in $\Lambda(\mathbb{R}^n)$, δ is an adjoint operator of exterior differentiation d, *i.e.*, we have

$$(d\alpha, \beta) = (\alpha, \delta\beta). \tag{2.5}$$

Conversely, d is an adjoint operator to δ.

3. Homotopy Decomposition. The homotopy decomposition approach considered in the present paper is based on the discussion in [3, Chapter 5]. The homotopy operator $\mathbb{H}$ is a linear operator on elements of $\Lambda^k(\mathbb{R}^n)$ that satisfies the identity

$$\omega = d(\mathbb{H}\omega) + \mathbb{H}d\omega, \tag{3.1}$$

for a given differential form $\omega \in \Lambda^k(\mathbb{R}^n)$. The first step in the construction of a homotopy operator is to define a star-shaped domain on $\mathbb{R}^n$. An open subset S of $\mathbb{R}^n$ is said to be star-shaped with respect to a point $\boldsymbol{p}^* = (x_1^*, \ldots, x_n^*) \in S$ if the following conditions hold:

- S is contained in a coordinate neighborhood U of $\boldsymbol{p}^*$;
- The coordinate functions of U assign coordinates $(x_1^*, \ldots, x_n^*)$ to $\boldsymbol{p}^*$;
- If p is any point in S with coordinates $(x_1, \ldots, x_n)$ assigned by functions of U, then the set of points $(x^* + \lambda(x - x^*))$ belongs to S, for all $\lambda \in [0, 1]$.

A star-shaped region S has a natural associated vector field $\mathfrak{X}$, defined in local coordinates by

$$\mathfrak{X}(x) = (x_i - x_i^*)\frac{\partial}{\partial x_i}, \quad \forall x \in S.$$

For a differential form ω of degree k on a star-shaped region S centered at the origin, the homotopy operator is defined, in coordinates, as

$$(\mathbb{H}\omega)(x) = \int_0^1 \mathfrak{X}(x^* + \lambda(x - x^*)) \lrcorner \omega(x^* + \lambda(x - x^*))\lambda^{k-1} d\lambda, \tag{3.2}$$

where $\omega(x^* + \lambda(x - x^*))$ denotes the differential form evaluated on the star-shaped domain in the local coordinates defined above. The important properties of the homotopy operator used in the present context are the following [3]:

1. $\mathbb{H}$ maps $\Lambda^k(S)$ into $\Lambda^{k-1}(S)$ for $k \geq 1$ and maps $\Lambda^0(S)$ identically to zero;
2. $d\mathbb{H} + \mathbb{H}d =$ identity for $k \geq 1$ and $(\mathbb{H}df)(x) = f(x) - f(x_0)$ for $k = 0$;
3. $(\mathbb{H}\mathbb{H}\omega)(x_i) = 0, \quad (\mathbb{H}\omega)(x_i^*) = 0$;
4. $\mathfrak{X} \lrcorner \mathbb{H} = 0, \quad \mathbb{H}\mathfrak{X}\lrcorner = 0$.

The first part of the right hand side of (3.1), $d(\mathbb{H}\omega)$, is obviously a closed form, since by definition of the exterior derivative, $d \circ d(\mathbb{H}\omega) = 0$. By the first property of the homotopy operator, for $\omega \in \Lambda^k(S)$, we have $(\mathbb{H}\omega) \in \Lambda^{k-1}(S)$, hence $d(\mathbb{H}\omega)$ is also exact on S. We denote the exact part of ω by $\omega_e = d(\mathbb{H}\omega)$ and the anti-exact part by $\omega_a = \mathbb{H}d\omega$.

4. Dual Homotopy Decomposition. In a manner that is similar to the definition of the homotopy operator, one can define a dual homotopy operator based on the notion of coexact forms. The general idea of the present paper is to identify that structure by using a decomposition that identifies the coexact and co-antiexact part of a given differential form. To the best of our knowledge, such an operator has never been described in the literature. The proposed dual homotopy operator $\mathbb{S}$ is a linear operator on elements of $\Lambda^k(\mathbb{R}^n)$ that satisfies the identity

$$\omega = \delta(\mathbb{S}\omega) + \mathbb{S}\delta\omega, \tag{4.1}$$

for a given differential form $\omega \in \Lambda^k(\mathbb{R}^n)$.

Proposition 4.1. *The dual homotopy operator can be written in terms of the homotopy as follows:*

$$\mathbb{S} = (-1)^{n(k+1)+1} \star \mathbb{H} \star \tag{4.2}$$

Proof. Left and right action of the Hodge operator on the identity

$$\mathbb{S} = (-1)^{n(k+1)+1} \star \mathbb{H} \star$$

yields

$$\star \mathbb{S} \star = (-1)^{n(k+1)+1} \star \star \mathbb{H} \star \star = (-1)^{n(k+1)+1} \mathbb{H} \tag{4.3}$$

Substitution in the identity

$$d\mathbb{H} + \mathbb{H}d = \mathbb{I}\mathrm{d}$$

one gets,

$$(-1)^{n(k+1)+1} d \star \mathbb{S} \star + (-1)^{n(k+1)+1} \star \mathbb{S} \star d = \mathbb{I}\mathrm{d}$$

Left and right action of the Hodge operator yields:

$$(-1)^{n(k+1)+1} \star d \star \mathbb{S} \star \star + (-1)^{n(k+1)+1} \star \star \mathbb{S} \star d \star = \star\star$$

By definition of the codifferential one obtains:

$$(-1)^{n(k+1)+1}(-1)^{-n(k+1)-1} \delta \mathbb{S} \star \star + \star \star \mathbb{S} (-1)^{n(k+1)+1}(-1)^{-n(k+1)-1} \delta = \star\star$$

Finally, using the property of the Hodge operator $\star\star = (-1)^{k(n-k)}$ and dividing both sides by $(-1)^{k(n-k)}$:

$$\delta \mathbb{S} + \mathbb{S} \delta = \mathbb{I}\mathrm{d}$$

as required. □ Using the identity (4.2), one can define a suitable integral version of the dual-homotopy operator as:

$$(\mathbb{S}\omega)(x) = (-1)^{n(k+1)+1} \star \int_0^1 \mathfrak{X}(x^* + \lambda(x - x^*)) \lrcorner \star \omega(x^* + \lambda(x - x^*)) \lambda^{k-1} d\lambda \tag{4.4}$$

For the special case of a one-form $\omega \in \Lambda^1$, one can compute an alternative expression for the dual homotopy operator. Note that a star-shaped domain S has a natural associated one-form $\mathfrak{W}$, defined in local coordinates by

$$\mathfrak{W}(x) = (x_i - x_i^*) dx_i, \quad \forall x \in S. \tag{4.5}$$

This one-form can be constructed constructed in a manner totally similar to the natural vector field $\mathfrak{X}$ on S.

Proposition 4.2. *For a differential one- form ω on a star-shaped region S centered at x^*, the dual homotopy operator is defined, in coordinates, as*

$$(\mathbb{S}\omega)(x) = \int_0^1 \mathfrak{W}(x^* + \lambda(x - x^*)) \wedge \omega(x^* + \lambda(x - x^*)) \lambda^{k-1} d\lambda, \tag{4.6}$$

where $\omega(x^* + \lambda(x - x^*))$ denotes the differential form evaluated on the star-shaped domain in the local coordinates.

Before closing this section, we provide a theorem, to be proved elsewhere, that summarizes some useful properties of the dual homotopy operator.

Theorem 4.3. *The dual homotopy operator has the following properties:*

1. $\mathbb{S}$ *maps* $\Lambda^k(S)$ *into* $\Lambda^{k+1}(S)$ *for* $k \geq 0$ *and maps* $\Lambda^n(S)$ *to zero;*
2. $\delta\mathbb{S} + \mathbb{S}\delta =$ *identity for* $0 \leq k < n-1$ *and* $(\mathbb{S}\delta\omega)(x) = f(x) - f(x_0)$ *for* $\omega \in \Lambda^{n-1}(S)$ *and* $f(x) \in \Lambda^0(S)$*;*
3. $(\mathbb{S}\mathbb{S}\omega)(x_i) = 0, \quad (\mathbb{S}\omega)(x_i^0) = 0$*;*
4. $\mathfrak{W} \wedge \mathbb{S} = 0, \quad \mathbb{S}\mathfrak{W}\wedge = 0.$

5. Application to Power-shaping. We illustrate how to use the decomposition of the one-form $\omega(x) = \star(f(x) \lrcorner \mu)$ derived in Section 2, using the homotopy operator and a dual homotopy operator, to study the problem of Brayton–Moser representation of (1.1) [4]. We first define a non vanishing closed two-form $\Omega = \sum_{1\leq i<j\leq n} dx_i \wedge dx_j$ on $\mathbb{R}^n$. The exterior derivative of $\omega(x)$ computed in (2.3) is given by

$$d(\omega) = \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} -\frac{\partial f_i}{\partial x_j} dx_j \wedge dx_i. \tag{5.1}$$

To compute the codifferential of ω, we first compute

$$\star\omega = \sum_{i=1}^{n} (-1)^{i+1} f_i \Lambda_i, \tag{5.2}$$

where

$$\Lambda_i = \begin{cases} dx_2 \wedge \ldots \wedge dx_n & \text{if } i = 1 \\ dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n & \text{if } i > 1 \end{cases}$$

The exterior derivative is given by

$$d(\star\omega) = \sum_{i=1} (-1)^{i+1} \frac{\partial f_i}{\partial x_i} dx_i \wedge \Lambda_i = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \mu. \tag{5.3}$$

Finally, taking the Hodge Star, we obtain the function

$$\delta\omega = \star d \star \omega = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}. \tag{5.4}$$

By applying the homotopy operator, $\omega = d(\mathbb{H}\omega) + \mathbb{H}d\omega$, we obtain a decomposition of the one-form ω in terms of an exact component and an anti-exact components, denoted by $\omega_e = d(\mathbb{H}\omega)$ and $\omega_a = \mathbb{H}d\omega$, respectively. Since, by the properties of exterior derivative that $dd = 0$ the exact part ω_e is closed and exact (i.e., ω_e is the exterior derivative of a 0-form, the function $\mathbb{H}\omega$). The dual homotopy operator, $\omega = \delta(\mathbb{S}\omega) + \mathbb{S}\delta\omega$, which decomposes the one-form ω in terms of a coexact component and an anti-coexact components, denoted by $\omega_c = \delta(\mathbb{S}\omega)$ and $\omega_s = \mathbb{S}\delta\omega$, respectively. As stated above, properties of the codifferential are such that $\delta\delta = 0$ the coexact part ω_c is coexact (i.e., ω_c is the codifferential of a the two-form $\mathbb{S}\omega$). Note that if ω is itself coexact, then we get that $\omega = \omega_c$.

Consider the nonlinear dynamical system given by

$$\dot{x} = -\bar{Q}(x) \frac{\partial P^T(x)}{\partial x} \tag{5.5}$$

where $\bar{Q}(x)$ is the pointwise matrix inverse of a matrix $Q(x)$, assumed in power-shaping to be symmetric and nonsingular [4]. We make a similar assumption in the following development. Consider the development in Section 2, for a modified vector field $\tilde{X}$ given by

$$\tilde{X} = \sum_{i=1}^{n} \Gamma(x) f(x) \frac{\partial}{\partial x_i}, \tag{5.6}$$

where $\Gamma(x)$ is a square symmetric nonsingular matrix-valued function. We first compute the following $(n-1)$ form

$$\tilde{j} = \tilde{X} \lrcorner \mu = \sum_{i=1}^{n} \tilde{f}_i(x)(-1)^{i+1}\Lambda_i.$$

Using the Hodge star operator, one obtains the one-form

$$\star(\tilde{j}) = \sum_{i=1}^{n} -\tilde{f}_i(x)dx_i = -\tilde{f}^T(x)dx = -f^T(x)\Gamma^T(x)dx.$$

Substituting for $f(x)$, one obtains:

$$\star(\tilde{j}) = -\frac{\partial P(x)}{\partial x}\bar{Q}^T(x)\Gamma^T(x)dx. \tag{5.7}$$

Choosing $\Gamma(x) = \bar{Q}^{-1}(x)$ yields to the exact form $\omega = -\frac{\partial P}{\partial x}dx$, *i.e.*, $d\omega = d(\star(\tilde{j})) = 0$. More generally, for a given desired structure $\tilde{X}$, one needs to obtain conditions under which such a matrix $\Gamma(x)$ exists. By homotopy decomposition, a potential for the system is given by $\psi(x) = \mathbb{H}\omega$. Since $d\mathbb{H}\omega = 0$, one obtains the structure encode in the anti-exact part by computing the exterior derivative of $\star(\tilde{j})$, which yields to the following two-form:

$$d(\star(\tilde{j})) = dx \wedge \left(f^T(x)\frac{\partial \Gamma^T(x)}{\partial x} + \frac{\partial f^T(x)}{\partial x}\Gamma(x)\right)dx. \tag{5.8}$$

The standard integrability condition [7] requires that the matrix

$$\Theta(x) = \Gamma\frac{\partial f(x)}{\partial x} + \frac{\partial \Gamma(x)}{\partial x}f(x)$$

is such that

$$\Theta_{ij}(x) = \Theta_{ji}(x) \tag{5.9}$$

for $i, j = 1, \ldots, n,\ i \neq j$. Taking the Hodge Star of $d(\star(\tilde{j}))$ one obtains the $(n-2)$ form,

$$\star d \star (\tilde{j}) = \sigma(I, J)\sum_{i=1}^{n}\sum_{j=1}^{n}(\Theta_{ji}\Lambda_{ji}). \tag{5.10}$$

where $\sigma(I, J)$ is the permutation function associated with the $(n-2)$ form Λ_{ji} given by

$$\Lambda_{ji} = dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n.$$

Vanishing of this $(n-2)$ form requires exactly the condition (5.9). Hence, the structure of the matrix $\Gamma(x)$ can be recovered from the anti-exact part, subjected to the symmetry condition on $\Theta(x)$, leading to conditions to express the original dynamics into the Brayton–Moser form.

Conversely, information about the systems dynamics can also be extracted from the codifferential of $\tilde{j}$, given by:

$$\delta\tilde{j} = \sigma(I, J)\sum_{i=1}^{n}\sum_{j=1}^{n}(\Theta_{ji}\Lambda_{ji}), \quad \Theta(x) = \bar{Q}(x)\frac{\partial^2 P(x)}{\partial x \partial x^T} + \frac{\partial \bar{Q}(x)}{\partial x}\frac{\partial P(x)}{\partial x}. \tag{5.11}$$

One can see that the first element in $\Theta(x)$ vanishes identically and therefore one obtains:

$$\delta j = \sigma(I, J) \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\sum_{k=1}^{n} \bar{Q}_{jk}(x) \frac{\partial^2 P(x)}{\partial x_k \partial x_i} + \frac{\partial \bar{Q}_{jk}(x)}{\partial x_i} \frac{\partial P(x)}{\partial x_k} \right) \Lambda_{ji}. \tag{5.12}$$

We can rewrite the matrix $\Theta(x)$ as:

$$\Theta(x) = \bar{Q}(x) \frac{\partial^2 P(x)}{\partial x \partial x^T} - \bar{Q}(x) \frac{\partial Q(x)}{\partial x} \bar{Q}(x) \frac{\partial P(x)}{\partial x}.$$

The potential for the system can thus be computed using $\psi(x) = \star \mathbb{S} \tilde{j}$, and the structure conditions for the Brayton–Moser realization is given by the vanishing of the two-form $\star \delta \tilde{j}$, in a way that mimic the homotopy operator approach.

6. Conclusion. This paper considers the problem of representing a sufficiently smooth nonlinear dynamical as a potential-driven system. The approach proposed in the present note is based on a decomposition of a differential one-form associated to the given vector field into its exact and anti-exact components, and into its coexact and anti-coexact components. In particular, a dual operator to the standard homotopy operator was defined and used in the present paper to invert the codifferential operator. Future research would seek to use the proposed approach to combine both operators to the representation of systems based on two potentials [9].

REFERENCES

[1] D. Cheng, T. Shen, and T.J. Tarn. Pseudo-Hamiltonian realization and its application. *Communications in Information and Systems*, 2(2):91–120, 2002.

[2] J. Cortés, A. van der Schaft, and P.E. Crouch. Characterization of gradient control systems. *SIAM Journal on Control and Optimization*, 44(4):1192–1214, 2005.

[3] D.G.B. Edelen. *Applied Exterior Calculus.* Dover Publications Inc., Mineola, NY, 2005.

[4] A. Favache, D. Dochain, and J.J. Winkin. Power-shaping control: Writing the system dynamics into the Brayton–Moser form. *Systems and Control Letters*, 60:618–624, 2011.

[5] E. García-Canseco, D. Jeltsema, R. Ortega, and J.M.A. Scherpen. Power-based control of physical systems. *Automatica*, 46:127–132, 2010.

[6] N. Hudon, K. Höffner, and M. Guay. Equivalence to dissipative Hamiltonian realization. In *Proceedings of the 47th IEEE Conference on Decision and Control*, pages 3163–3168, Cancun, Mexico, 2008.

[7] J.M. Lee. *Introduction to Smooth Manifolds*, volume 218 of *Graduate Texts in Mathematics.* Springer, New York, NY, 2006.

[8] S. Morita. *Geometry of Differential Forms*, volume 201 of *Translations of Mathematical Monographs.* American Mathematical Society, Providence, RI, 2001.

[9] W.-H. Steeb and T. Scholes. Nonlinear dynamical systems and decomposition of vector fields. *Physica Scripta*, 72:16–18, 2005.

[10] S.L. Yap. The Poincaré lemma and an elementary construction of vector potentials. *The American Mathematical Monthly*, 116(3):261–267, 2009.