STABILIZATION OF SWITCHED LINEAR STOCHASTIC SYSTEMS UNDER DELAYED DISCRETE MODE OBSERVATIONS *

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Abstract. Almost sure asymptotic stabilization of switched linear stochastic systems is investigated. A control framework is developed for the case where the mode of the switched system is periodically sampled at discrete time instants and the obtained sampled mode information is subject to time delays. Based on our probabilistic analysis of a bivariate stochastic process composed of the actual mode signal and its delayed sampled version, we derive sufficient conditions under which our proposed control law guarantees stochastic stability of the zero solution of the closed-loop system.

Key words. switched stochastic systems, sampled mode information, delayed mode information

AMS subject classifications. 93C30, 93E03, 93C57

1. Introduction.

Stochastic hybrid systems provide accurate models for various real-life processes from finance, physics and engineering fields. Recently, considerable attention has been devoted on the stabilization of stochastic hybrid systems. In particular, feedback control of Markov jump systems has been extensively studied (e.g., [2, 5, 6], and the references therein). Markov jump systems are composed of deterministic subsystems (modes) and a stochastic mode signal that is modeled as a finite-state Markov chain. Furthermore, researchers have also explored more general "switching diffusion processes", which incorporate stochastic subsystem dynamics and a stochastic mode signal. Specifically, feedback control of "switching diffusion processes" is discussed in [8, 12, 20, 21].

Most of the feedback control laws documented in the literature of stochastic hybrid systems require perfect knowledge of the mode signal. As a consequence, when the mode of the switched system is not available for control purposes or only sampled mode information is available, these control laws cannot be used for stabilization. It is therefore important to develop feedback control frameworks for the limited mode information case. In our previous work [3, 4], we explored the case where the mode information is observed (sampled) only at discrete time instants. We proposed feedback control laws for stabilizing continuous-time switched linear stochastic dynamical systems under sampled mode information. In this present study, we extend our previous results presented in [4] to the case where the sampled mode information is subject to time delay.

A feedback control problem for stochastic hybrid systems under the effect of delays is explored in several studies. Specifically, stabilization with delayed state feedback has been investigated in [1, 7, 9, 15, 19] and stabilization of discrete-time Markov jump systems over communication networks with delays has been discussed in [10, 18]. Furthermore, an optimal controller is obtained in [13] for stabilizing discrete-time linear Markov jump systems under "one time-step" delayed mode observations.

In this paper, we investigate the feedback control problem for continuous-time switched stochastic systems under sampled and delayed mode information. These

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systems are composed of linear stochastic subsystems, which include Brownian motion in their dynamics. A continuous-time, finite-state Markov chain is employed for modeling the mode signal of the switched system. We focus on the case where the mode signal of the switched system is observed (sampled) only at equally spaced discrete time instants and the obtained mode samples are available to the controller only after a time delay. The mode information time delay can capture communication delays between the mode sampling mechanism and the controller. On the other hand, computational delays in mode detection might as well be modeled by the delayed mode observations. For example, mode information delays may correspond to failure-detection delays for a fault tolerant control system with normal/faulty modes and a "fault detection and isolation scheme" explored in [11, 17]. We propose a piecewise-continuous control law that depends only on the delayed version of the sampled mode signal rather than the actual mode signal. We employ a quadratic Lyapunov-like function to obtain sufficient conditions under which our proposed control law guarantees almost sure asymptotic stability of the switched stochastic system.

The paper is organized as follows. In Section 2, we explain the notation used in the paper; in addition, we provide a review of Markov chains, and the definition of almost sure asymptotic stability. We explain the feedback control problem for switched stochastic systems under sampled and delayed mode information in Section 3, where we also obtain sufficient conditions under which our proposed control law guarantees almost sure asymptotic stability. In Section 4, we give an illustrative numerical example. Finally, we conclude the paper in Section 5.

2. Mathematical Preliminaries. In this section we introduce notation and several definitions concerning stochastic dynamical systems. Throughout the paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices, $\mathbb{N}$ and $\mathbb{N}_0$ respectively denote the set of positive and nonnegative integers. Furthermore, we write $(\cdot)^{\mathrm{T}}$ for transpose and $\mathrm{tr}(\cdot)$ for trace of a matrix, I_n for the identity matrix of dimension n, and $\lambda_{\min}(H)$ (resp., $\lambda_{\max}(H)$) for the minimum (resp., maximum) eigenvalue of the Hermitian matrix H.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A filtration $\{\mathcal{F}_t\}_{t\geq 0}$ on this probability space is a family of σ-algebras such that

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}, \quad 0 \leq s < t.$$

A stochastic process $\{x_t(\omega)\}_{t\geq 0}$ is said to be adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if the random variable $x_t : \Omega \to \mathbb{R}^n$ is $\mathcal{F}_t$-measurable, that is,

$$\{\omega \in \Omega : x_t(\omega) \in B\} \in \mathcal{F}_t, \quad t \geq 0,$$

for all Borel sets $B \subset \mathbb{R}^n$. The notations $\mathbb{P}[\cdot]$ and $\mathbb{E}[\cdot]$ denote the probability and expectation, respectively.

A finite-state, continuous-time Markov chain is a piecewise-constant and right-continuous stochastic process that takes values from a finite set $\mathcal{M} \triangleq \{1, 2, \ldots, M\}$. Mathematically, a Markov chain is defined to be an $\mathcal{F}_t$-adapted stochastic process $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ characterized by a generator matrix $Q \in \mathbb{R}^{M\times M}$. The rates of transition between each pair of states $i, j \in \mathcal{M}$ are given by

$$\mathbb{P}[r(t+\Delta t) = j | r(t) = i] = \begin{cases} q_{i,j}\Delta t + o(\Delta t), & i \neq j, \\ 1 + q_{i,j}\Delta t + o(\Delta t), & i = j, \end{cases} \tag{2.1}$$

where $q_{i,j}$ denotes the (i,j)th element of the generator matrix Q. Note that $q_{i,j} \geq 0$, $i \neq j$, and $q_{i,i} = -\sum_{j \neq i} q_{i,j}$, $i \in \mathcal{M}$. A Markov chain is called "irreducible" if it is possible to reach from any state to another state with one or more transitions. For every finite-state, irreducible Markov chain there exists a unique stationary probability distribution $\pi \triangleq [\pi_1, \ldots, \pi_M]^{\mathrm{T}} \in \mathbb{R}^M$ such that $\pi^{\mathrm{T}} Q = 0$, $\pi_i > 0$, $i \in \mathcal{M}$, and $\sum_{i \in I} \pi_i = 1$ [14, 16]. We employ a finite-state Markov chain for modeling the mode signal, which manages the transition between subsystems (modes) of the switched system.

In this paper, we adopt almost sure asymptotic stability notion, which is also called "asymptotic stability with probability one" [12]. Specifically, the zero solution $x(t) \equiv 0$ of a stochastic dynamical system is asymptotically stable almost surely if

$$\mathbb{P}\{\omega \in \Omega : \lim_{t \to \infty} \|x_t(\omega)\| = 0\} = 1, \tag{2.2}$$

for $t \geq 0$ with a fixed initial condition $x_0(\cdot)$.

3. Main Results. We consider the continuous-time switched stochastic system with $M > 0$ modes given by

$$\mathrm{d}x(t) = A_{r(t)} x(t) \mathrm{d}t + B_{r(t)} u(t) \mathrm{d}t + D_{r(t)} x(t) \mathrm{d}W(t), \tag{3.1}$$

with the initial conditions $x(0) = x_0$ and $r(0) = r_0$, where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ respectively denote the state vector and the control input, $\{W(t) \in \mathbb{R}\}_{t \geq 0}$ is an $\mathcal{F}_t$-adapted Wiener process, A_i, $D_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, $i \in \mathcal{M} \triangleq \{1, 2, \ldots, M\}$, are subsystem matrices. The mode signal $\{r(t) \in \mathcal{M}\}_{t \geq 0}$ is assumed to be an $\mathcal{F}_t$-adapted irreducible Markov chain characterized by the generator matrix $Q \in \mathbb{R}^{M \times M}$ with the stationary probability distribution $\pi \in \mathbb{R}^M$. The Wiener process $\{W(t) \in \mathbb{R}\}_{t \geq 0}$ and the mode signal $\{r(t) \in \mathcal{M}\}_{t \geq 0}$ are assumed to be mutually independent stochastic processes.

In the following, we investigate the feedback control problem for the case where the mode signal is sampled and the obtained sampled mode information is subject to time delay. Specifically, the mode signal is assumed to be observed (sampled) periodically with period $\tau > 0$. Moreover, the obtained mode samples are assumed to be available to the controller after a constant time delay $T_{\mathrm{D}} > 0$.

We denote the available mode samples by the sequence $\{r(k\tau) \in \mathcal{M}\}_{k \in \mathbb{N}_0}$. By employing the "sample and hold" technique we obtain the sampled version of the mode signal $\{\sigma(t) \in \mathcal{M}\}_{t \geq 0}$ defined by

$$\sigma(t) \triangleq r(k\tau), \quad t \in [k\tau, (k+1)\tau), \quad k \in \mathbb{N}_0. \tag{3.2}$$

In our previous study [4], we had considered a control law of the form $u(t) = K_{\sigma(t)} x(t)$, which depends only on the sampled mode information. In this present study, each mode sample data is assumed to be subject to delay $T_{\mathrm{D}} > 0$, and hence only a delayed version of the sampled mode signal $\{\sigma(t) \in \mathcal{M}\}_{t \geq 0}$ is available for control purposes. As a result, the control law presented in [4] cannot be directly employed. We assume that the initial mode is known to the controller and propose a new control law of the form

$$u(t) = \begin{cases} K_{r_0} x(t), & 0 \leq t < T_{\mathrm{D}}, \\ K_{\sigma(t - T_{\mathrm{D}})} x(t), & t \geq T_{\mathrm{D}}, \end{cases} \tag{3.3}$$

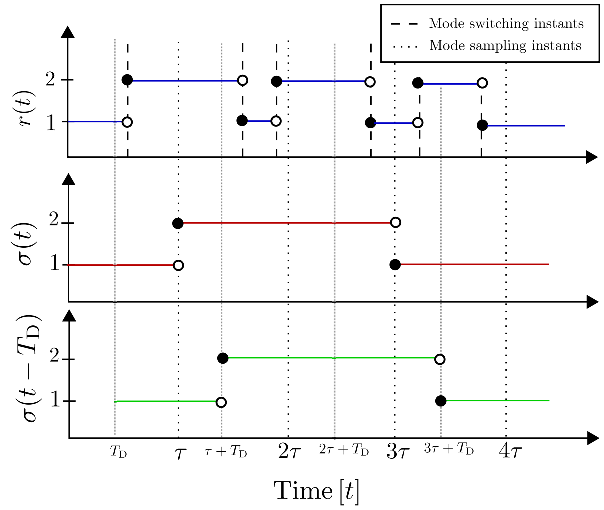

FIG. 3.1. *Actual mode signal $r(t)$, the sampled mode signal $\sigma(t)$, and the delayed version of the sampled mode signal $\sigma(t-T_{\rm D})$ versus time*

which depends only on the delayed version of the sampled mode signal. Henceforth, our main objective is to obtain sufficient conditions of almost sure asymptotic stability of the closed-loop system (3.1) under our proposed control law (3.3).

The delayed sampled mode signal $\{\sigma(t-T_{\rm D})\in\mathcal{M}\}_{t\geq T_{\rm D}}$ is a piecewise-constant stochastic process that depends on the actual mode signal $\{r(t)\in\mathcal{M}\}_{t\geq 0}$. Note that the frequency of the occurrences of mode transitions, mode sampling period $\tau>0$, and sampled mode information delay $T_{\rm D}>0$ affect how accurately the actual mode signal $\{r(t)\in\mathcal{M}\}_{t\geq 0}$ is represented by the delayed sampled version $\{\sigma(t-T_{\rm D})\in\mathcal{M}\}_{t\geq T_{\rm D}}$. Fig. 3.1 shows sample paths of $r(t)$, $\sigma(t)$, and $\sigma(t-T_{\rm D})$ of an example switched system (3.1) with $M=2$ modes.

3.1. Sufficient conditions for almost sure asymptotic stabilization. We employ a quadratic Lyapunov approach similar to the one used in [3,4] to obtain sufficient conditions of almost sure asymptotic stabilization under sampled and delayed mode information.

THEOREM 3.1. *Consider the switched linear stochastic system (3.1) with mode sampling period $\tau>0$ and sampled mode information constant time delay $T_{\rm D}>0$. If there exist $P>0$ and scalars $\zeta_i\in\mathbb{R}$, $i\in\mathcal{M}$, such that*

$$0\geq A_i^{\rm T}P+PA_i+D_i^{\rm T}PD_i-2PB_iB_i^{\rm T}P-\zeta_iP, \tag{3.4}$$

for $i\in\mathcal{M}$, and

$$\frac{1}{\tau}{\rm tr}(\Pi\int_{T_{\rm D}}^{T_{\rm D}+\tau}e^{Qs}{\rm d}s\Gamma)-\sum_{i\in\mathcal{M}}\pi_i\frac{\lambda_{\min}^2(D_i^{\rm T}P+PD_i)}{2\lambda_{\max}^2(P)}<0, \tag{3.5}$$

where $\Pi \in \mathbb{R}^{M\times M}$ *denotes the diagonal matrix with the diagonal entries* $\pi_1, \pi_2, \ldots, \pi_M$*, and* $\Gamma \in \mathbb{R}^{M\times M}$ *denotes the matrix with the* (i,j)*th entries given by*

$$\gamma_{i,j} = \begin{cases} \zeta_j, & i = j, \\ \zeta_i + \frac{2\lambda_{\max}(PB_iB_i^{\mathrm{T}}P)}{\lambda_{\min}(P)} \\ \qquad -\frac{\lambda_{\min}(P(B_jB_i^{\mathrm{T}}+B_iB_j^{\mathrm{T}})P)}{\lambda_{\max}(P)}, & i \neq j, \end{cases} \tag{3.6}$$

then the feedback control law (3.3) with the feedback gain matrices given by

$$K_i = -B_i^{\mathrm{T}}P, \quad i \in \mathcal{M}, \tag{3.7}$$

guarantees that the zero solution $x(t) \equiv 0$ *of the closed-loop system (3.1) and (3.3) is asymptotically stable almost surely.*

Note that the stabilization conditions (3.4) and (3.5) depend not only on the subsystem dynamics, but also on the probabilistic dynamics of the mode signal, as well as the mode sampling period $\tau > 0$ and the sampled mode information delay $T_{\mathrm{D}} > 0$.

When time delay for the sampled mode information T_{D} tends to 0, the conditions of Theorem 3.1 reduces to the stabilization conditions presented in [4] for the case where time delay for sampled mode information is not present.

Note that under the conditions (3.4) and (3.5), our proposed control law (3.3) guarantees almost sure stabilization even if the mode samples are subject to different time delays that are upper-bounded by a constant $T_{\mathrm{D}} \in (0, \tau]$. Specifically, consider the case where the kth mode sample data $r(k\tau)$ is subject to time delay $T_k > 0$, $k \in \mathbb{N}_0$. In this case, the sequence $\{k\tau + T_k\}_{k\in\mathbb{N}_0}$ denotes the time instants at which the obtained mode samples become available to the controller. If there exists a constant $T_{\mathrm{D}} \in (0, \tau]$ such that $T_k \in (0, T_{\mathrm{D}}]$, $k \in \mathbb{N}_0$, then the mode samples become available to the controller *in order*, that is,

$$k_1\tau + T_{k_1} \leq k_2\tau + T_{k_2}, \quad k_1 \leq k_2, \quad k_1, k_2 \in \mathbb{N}_0. \tag{3.8}$$

Furthermore,

$$k\tau + T_k \leq k\tau + T_{\mathrm{D}}, \quad k \in \mathbb{N}_0, \tag{3.9}$$

which implies that the controller has the sampled mode information $r(k\tau)$ at time $k\tau + T_{\mathrm{D}}$, and hence the proposed control law (3.3) can still be employed for stabilizing the switched linear stochastic dynamical system (3.1).

4. Numerical Example. In this section, we demonstrate the efficacy of our approach concerned with the stabilization of switched linear stochastic dynamical systems under sampled and delayed mode information. Specifically, we consider the switched linear stochastic system (3.1) composed of $M = 3$ subsystems characterized by the subsystem matrices

$$A_1 = \begin{bmatrix} 1 & -5 \\ 1 & 1.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -1.6 \\ 1.6 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.5 & 0 \\ 0.75 & 0.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 0.5 & 0.5 \\ -4.75 & 0.5 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 2 \\ -2 \end{bmatrix},$$

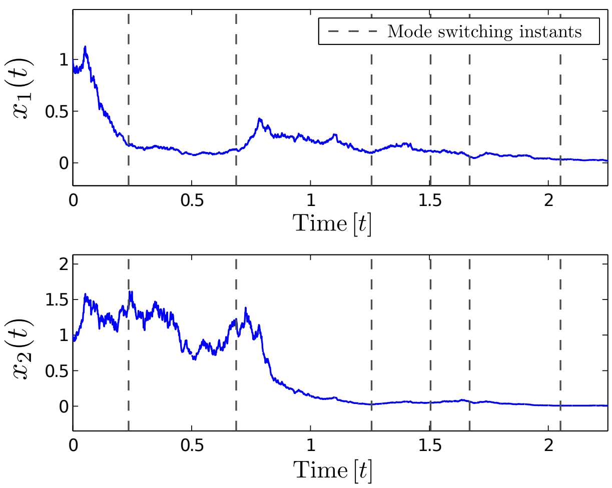

Fig. 4.1. *State trajectory versus time*

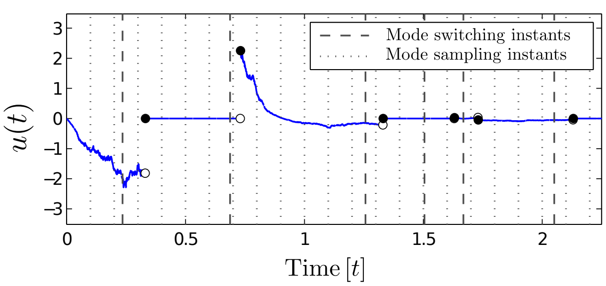

Fig. 4.2. *Control input versus time*

and $D_1 = D_2 = D_3 = I_2$. The mode signal $\{r(t) \in \mathcal{M} \triangleq \{1, 2, 3\}\}_{t\geq 0}$ of the switched system is assumed to be a Markov chain with the generator matrix

$$Q = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \tag{4.1}$$

with stationary probability distributions, $\pi_i = \frac{1}{3}$, $i \in \mathcal{M}$. Furthermore, the mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ is assumed to be sampled periodically at the time instants $k\tau$, $k \in \mathbb{N}_0$, where $\tau = 0.1$ is the mode sampling period. Furthermore, the obtained mode samples are assumed to be available to the controller after a constant time delay $T_{\rm D} = 0.03$.

Note that the positive-definite matrix $P = I_2$ and the scalars $\zeta_1 = -0.35$, $\zeta_2 = 2.75$, $\zeta_3 = -2.25$ satisfy (3.4) and (3.5). Therefore, it follows from Theorem 3.1 that the zero solution $x(t) \equiv 0$ of the switched stochastic system given by (3.1) under the proposed control law (3.3) is asymptotically stable almost surely.

Figs. 4.1 and 4.2 respectively show the sample paths of $x(t)$ and $u(t)$ obtained with the initial conditions $x(0) = [1,\, 1]^{\rm T}$ and $r(0) = 1$. The sampled mode signal $\sigma(t)$ may change its value at mode sampling time instants denoted by the sequence

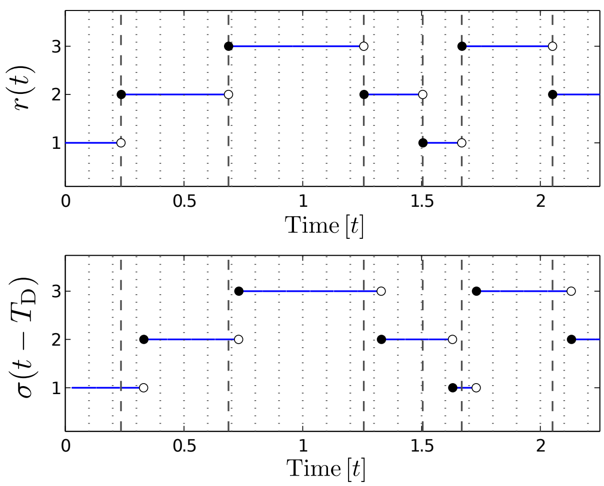

FIG. 4.3. *Actual mode signal $r(t)$ and the delayed version of the sampled mode signal $\sigma(t - T_D)$ versus time*

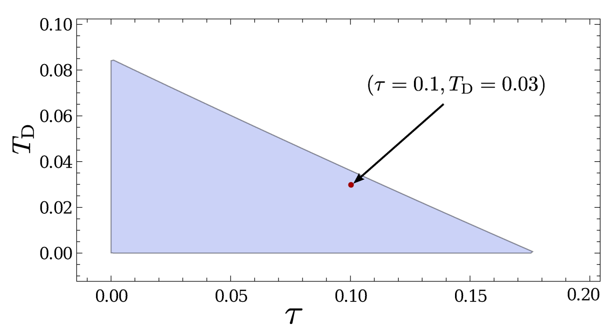

FIG. 4.4. *Stabilization region with respect to τ and T_D*

$\{k\tau\}_{k\in\mathbb{N}}$. Since, the piecewise-continuous control law (3.3) depends on the delayed sampled mode signal $\sigma(t - T_D)$, the control input trajectory is subject to jumps at time instants denoted by $\{k\tau + T_D\}_{k\in\mathbb{N}}$.

Note that the feedback control performance is directly related to the quality of the representation of the actual mode signal by the sampled and delayed version that is available to the controller. Owing to frequent mode sampling and small mode information time delay, the delayed sampled mode signal $\sigma(t - T_D)$ of this numerical example is a good representation of the actual mode signal $r(t)$ (see Fig. 4.3).

The stabilization depends on both the mode sampling period $\tau > 0$ and the mode sample information delay $T_D > 0$. Fig. 4.4 shows the numerically obtained values of the constants τ and T_D that satisfy the condition (3.5) of Theorem 3.1 for the positive-definite matrix $P = I_2$ and the scalars $\zeta_1 = -0.35$, $\zeta_2 = 2.75$, $\zeta_3 = -2.25$. The dark region represents the set of values $(\tau, T_D) \in (0, \infty) \times (0, \infty)$, for which the stabilization is guaranteed by the control law (3.3) according to Theorem 3.1. Note that the sample paths of $x(t)$, $u(t)$, $r(t)$ and $\sigma(t - T_D)$ shown in Figs. 4.1–4.3 are obtained for the values $\tau = 0.1$, $T_D = 0.03$, which corresponds to a point in the stabilization region shown in Fig. 4.4.

5. Conclusion. The stabilization of switched linear stochastic systems has been investigated. A feedback control framework has been developed for the case where the mode of the switched system is periodically sampled and the obtained mode samples are available to the controller only after a time delay. We employed a quadratic Lyapunov-like function for obtaining sufficient conditions under which our proposed control framework guarantees almost sure asymptotic stabilization.

REFERENCES

[1] E. K. Boukas, *Delay-dependent stability and output feedback stabilisation of Markov jump system with time-delay*, IEE Proc. Contr. Theo. Appl., 149 (2002), pp. 379–386.

[2] ———, *Static output feedback control for stochastic hybrid systems: Lmi approach*, Automatica, 42 (2006), pp. 183–188.

[3] A. Cetinkaya and T. Hayakawa, *Stabilization of switched linear stochastic dynamical systems under limited mode information*, in Proc. IEEE Conf. Dec. Contr., Orlando, FL, 2011, pp. 8032–8037.

[4] ———, *Feedback control of switched stochastic systems using uniformly sampled mode information*, in Proc. Amer. Contr. Conf. (in print), 2012.

[5] D. P. de Farias, J. C. Geromel, J. B. R. do Val, and O. L. V. Costa, *Output feedback control of Markov jump linear systems in continuous-time*, IEEE Trans. Autom. Contr., 45 (2000), pp. 944–949.

[6] Y. Fang and K. A. Loparo, *Stabilization of continous-time jump linear systems*, IEEE Trans. Autom. Contr., 47 (2002), pp. 1590–1602.

[7] Z. Fei, H. Gao, and P. Shi, *New results on stabilization of Markovian jump systems with time delay*, Automatica, 45 (2009), pp. 2300–2306.

[8] M. K. Ghosh, A. Arapostathis, and S. I. Marcus, *Optimal control of switching diffusions with application to flexible manufacturing systems*, SIAM J. Contr. Optm., 30 (1992), pp. 1–23.

[9] S. He and F. Liu, *On delay-dependent stability of Markov jump systems with distributed time-delays*, Circ. Sys. Sign. Proc., 30 (2011), pp. 323–337.

[10] M. Liu, D. Ho, and Y. Niu, *Stabilization of Markovian jump linear system over networks with random communication delay*, Automatica, 45 (2009), pp. 416–421.

[11] M. Mahmoud, J. Jiang, and Y. Zhang, *Analysis of the stochastic stability for fault tolerant control systems*, in Proc. IEEE Conf. Dec. Contr., Phoenix, AZ, 1999, pp. 3188–3193.

[12] X. Mao and C. Yuan, *Stochastic Differential Equations with Markovian Switching*, Imperial College Press: London, 2006.

[13] I. Matei, N. C. Martins, and J. S. Baras, *Optimal linear quadratic regulator for Markovian jump linear systems in the presence of one time-step delayed mode observations*, in Proc. 17th World. Cong. IFAC, Seoul, Korea, 2008, pp. 8056–8061.

[14] J.R. Norris, *Markov Chains*, Cambridge University Press: New York, 2009.

[15] S. Sathanantan, O.Adetona, C. Beane, and L. H. Keel, *Feedback stabilization of Markov jump linear system with time-varying delay*, Stoc. Anal. App., 26 (2008), pp. 577–594.

[16] R. Serfozo, *Basics of Applied Stochastic Processes*, Springer-Verlag: New York, 2009.

[17] F. Tao and Q. Zhao, *Synthesis of active fault-tolerant control based on Markovian jump system models*, IET Contr. Theo. App., 1 (2007), pp. 1160–1168.

[18] J. Wang and H. Yang, *Output-feedback stabilization of Markovian jump linear systems over networks with time delays*, J. Int. Inf. Comp. Sci., 13 (2011), pp. 2545–2552.

[19] W. Xie, G. Ma, C. Wang, and B. Feng, *Delay-dependent robust sampled-data stability for Markovian jump systems with time-delay*, in IEEE Conf. Ind. Elec. Appl., 2007.

[20] G. Yin and C. Zhu, *Hybrid Switching Diffusions: Properties and Applications*, Springer-Verlag: New York, 2010.

[21] C. Yuan and J. Lygeros, *On the exponential stability of switching diffusion process*, IEEE Trans. Autom. Contr., 50 (2005), pp. 1422–1426.