BALANCING AS A MOMENT MATCHING PROBLEM*

T.C. IONESCU†, J.M.A. SCHERPEN‡, O.V. IFTIME§, AND A. ASTOLFI¶

Abstract. In this paper, we treat a time-domain moment matching problem associated to balanced truncation. We show that in the class of parameterized reduced order models that achieve moment matching, there exists a subclass of models that have prescribed Hankel singular values. From this subclass, we can identify the balanced reduced order model, by solving an inverse moment matching problem. Finally, we show that the Hankel singular values are related to the moments of the system at the antistable eigenvalues of the system.

Key words. balancing, moment matching, Hankel singular values.

1. Introduction.

Balanced truncation is a widely popular method of model reduction for linear and nonlinear systems. In the systems and control literature, balancing was first introduced by Moore in 1981 [11] and consists of finding a change of coordinates such that, in the new coordinates, the controllability and observability Gramians of the given linear system are equal and diagonal. The diagonal elements are positive numbers, called Hankel singular values and represent, amongst others, the degree of controllability and observability of the corresponding state. Used for model reduction, the balanced realization is truncated by ignoring the dynamics corresponding to the lower singular values, i.e., the most non-minimal part. From a computational point of view, balanced truncation may be quite cumbersome, especially at very high dimensions, due to two major issues: the computation of the Gramians and the computation of the balancing change of coordinates. The former, are the solutions of two Lyapunov equations, whereas the latter may involve, for instance, in the nonlinear case, solving large nonlinear algebraic systems.

An alternative to balanced truncation is moment matching based model reduction, see e.g. [9, 4, 1, 13, 6, 10, 2] for a complete overview for linear systems. Using a numerical approach based on Krylov projection methods, the (reduced order) model is obtained by efficiently constructing a lower degree rational function that approximates a given transfer function (assumed rational). The low degree rational function matches the given transfer function at various points in the complex plane. A time-domain approach to moment matching has been taken in [3]. In short, the notion of moment of a linear, minimal system has been related to the unique solution of the Sylvester equation, see also, e.g., [8, 7], for previous results. Furthermore, the moments correspond to the well-defined steady-state response of the given system driven by a signal generator, which contains the interpolation points. Dually, the moments have been connected to the solution of a dual Sylvester equation, which further

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†Department of Electrical and Electronic Engineering, Imperial College London, SW7 2AZ, London, UK. E-mail: t.ionescu@imperial.ac.uk.

‡Department of Discrete Technology and Production Automation, Industrial Technology and Management, Fac. Math. & Natural Sciences, University of Groningen, Nijenborgh 4, 9747 AG, Groningen, The Netherlands (j.m.a.scherpen@rug.nl).

§Department of Economics, Econometrics and Finance, Faculty of Economics and Business, University of Groningen, Nettelbosje 2, 9747 AE, Groningen, The Netherlands (o.v.iftime@rug.nl).

¶Department of Electrical and Electronic Engineering, Imperial College London, SW7 2AZ, London, UK and with Dipartimento di Informatica, Sistemi e Produzione, Università di Roma Tor Vergata, 00133 Roma, Italy (a.astolfi@imperial.ac.uk).

correspond to the steady state response of the given system driving a (generalized) signal generator. Used for model reduction, the time-domain approach and its dual each, yield simple and direct characterizations of classes of parameterized, reduced order models that match a prescribed set of moments of a given system at a set of *finite* interpolation points. The classes of reduced order models that achieve moment matching contain subclasses of models that meet additional constraints, i.e., the free parameters are useful for enforcing properties such as, e.g., passivity, stability, relative degree, etc, irrespective of the choice of interpolation points.

In this paper we consider solving a time-domain moment matching problem associated to balanced truncation. First, from the class of models that achieve moment matching, we compute the (subclass) of reduced order models which have desired Hankel singular values. Then, in order to obtain the balanced truncated reduced order model, with specific significant Hankel singular values, we consider a converse moment matching problem. Given a linear, minimal, asymptotically stable system (A, B, C), with the Hankel singular values $\sigma_1 > ... > \sigma_n > 0$, find the (observable) pair (L, S) and the parameter G such that the reduced order model $(S - GL, G, C\Pi)$ of dimension ν, with Π the solution of the Sylvester equation $A\Pi + BL = \Pi S$, has the following properties

1. The reduced order model $(S - GL, G, C\Pi)$ is well defined, and matches the moments of the system (A, B, C) at $\sigma(S)$, i.e., $\sigma(S) \cap \sigma(A) = \emptyset$, $\sigma(S - GL) \cap \sigma(S) = \emptyset$.
2. If $\nu = n$, the model is a balanced representation of (A, B, C) with the Hankel singular values $\sigma_1, ..., \sigma_n$.
3. If $\nu < n$, the model is (an approximation of) a balanced truncation of the system (A, B, C), with the (approximate) Hankel singular values $\{\sigma_1, ..., \sigma_\nu\} \subset \{\sigma_1, ..., \sigma_n\}$.

Since the Hankel singular values are difficult to compute, we relate them to a moment matching problem. We do that by relating the Gramians of the (balanced) realizations with the solution of the Sylvester equation corresponding to a moment matching problem. We relate the moments of the given linear system, at specific interpolation points, i.e., the antistable eigenvalues, to the Hankel singular values of the system. Furthermore, we define a class of models that match these moments such that they provide information about the desired Hankel singular values.

The paper is organized as follows. In Section 2 we give a brief overview of both balanced truncation and moment matching techniques. In Section 3 we define the (subclass) of the reduced order models that achieve moment matching and have prescribed Hankel singular values. In Section 4 we solve the converse moment matching problem that allows us to obtain a balanced reduced order model that achieves moment matching. We solve the problem for second order systems. In Section 5 we relate the Hankel singular values to the moments of the system at the antistable modes. The paper ends with Conclusions.

2. Preliminaries. Consider the following linear, single input, single output, asymptotically stable, minimal system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$ and matrices A, B, C are real with appropriate dimensions. Consider the controllability Gramian W and the observability Gramian

M of the system (2.1). Since the system is assumed minimal and asymptotically stable, $W > 0$ and $M > 0$, and are the unique solutions of following the Lyapunov equations, respectively

$$AW + WA^T = BB^T, \tag{2.2}$$
$$A^T M + MA = C^T C. \tag{2.3}$$

Denote by $\sigma_1 > \sigma_2 > ... > \sigma_\nu \gg \sigma_{\nu+1} > ... > \sigma_n > 0$ the Hankel singular values of system (2.1). The system (A, B, C) is in balanced form if $W = M = \text{diag}\{\sigma_1, ..., \sigma_n\}$. Balancing the system (A, B, C) means finding a coordinate transformation, called balancing transformation, such that the system is brought into balanced form.

Let $S \in \mathbb{R}^\nu$, $\nu \leq n$ be such that $\lambda(S) \cap \lambda(A) = \emptyset$ and $L \in \mathbb{R}^{1\times\nu}$ be such that the pair (L, S) is observable. According to [3], the moments of system (2.1) at $\lambda(S)$ are given by $C\Pi$, where Π is the unique solution of the Sylvester equation

$$A\Pi + BL = \Pi S. \tag{2.4}$$

The class of models of order ν that match the moments of system (2.1) at $\lambda(S)$ is given by the following equations

$$\begin{aligned} \dot{\xi} &= (S - GL)\xi + Gu, \\ y &= C\Pi\xi, \end{aligned} \tag{2.5}$$

where $G \in \mathbb{R}^\nu$ are free parameters, such that $\lambda(S - GL) \cap \lambda(S) = \emptyset$.

3. Moment matching with prescribed Hankel singular values. Choosing G such that $\lambda(S - GL) \subset \mathbb{C}^-$ yields the subclass of reduced models of order ν which are asymptotically stable and minimal. From the subclass of asymptotically stable models we want to find the model(s) with the Hankel singular values $\sigma_1 > ... > \sigma_\nu > 0$.

PROPOSITION 3.1. *Let S be such that $0 \notin \lambda(S)$. Let system $(S - GL, G, C\Pi)$ be a model from the subclass of asymptotically stable, minimal models that match the moments of system (2.1) at $\lambda(S)$. Let $\hat{W}$ and $\hat{M}$ be the Gramians of $(S-GL, G, C\Pi)$. Then there exists G such that $\lambda(\hat{W}\hat{M}) = \{\sigma_1^2, ..., \sigma_\nu^2\}$, i.e. there exists a reduced order model $(S - GL, G, C\Pi)$ which matches the moments and inherits the more important Hankel singular values of system (2.1).* □

Example.

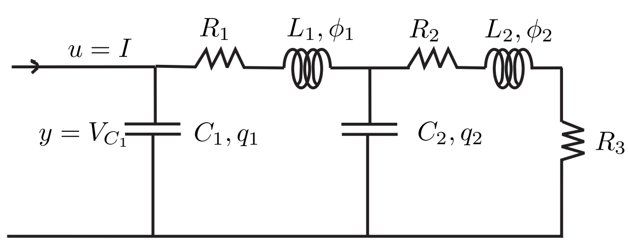

FIG. 3.1. *Ladder network - system of order 4.*

Consider the ladder network in Fig. 3.1, with $C_1 = L_1 = L_2 = R_1 = R_2 = R_3 = 1$ and $C_2 = 2$. A state space realization of this system is given by (2.1) with

$x = [q_1\ \phi_1\ q_2\ \phi_2]^T$ and

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & -2 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & -2 \end{bmatrix},\ C = [1\ 0\ 0\ 0] = B^T.$$

The Hankel singular values are $\sigma_1 = 1.4895 > \sigma_2 = 0.0608 > \sigma_3 = 0.0413 > \sigma_4 = 0.009$. Let $S = \mathrm{diag}\left\{\frac{1}{2}, 1\right\}$ and $L = [1,\ 1]$. The moments of (A, B, C) at $\lambda(S)$ are $\eta_0 = 1.2061$, $\eta_1 = 0.7619$. The class of reduced order models that match the moments η_0 and η_1 are given by (F, G, H) with

$$F = \begin{bmatrix} 0.5 - g_1 & -g_1 \\ -g_2 & 1 - g_2 \end{bmatrix},\ G = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix},\ H = [\eta_0\ \eta_1].$$

The subclass of minimal, asymptotically stable models is $(F, G, H)_{\mathbf{G}}$ characterized by $G \in \mathbf{G} = \left\{ \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \middle| g_1 \in \left(-\infty, -\frac{1}{2}\right),\ g_2 \in \left(\frac{3}{2} - g_1, 1 - 2g_1\right),\ g_2 \neq 1.358 - 1.583 g_1 \right\}$. In the subclass of asymptotically stable models $(F, G, H)_{\mathbf{G}}$, we find the model which has the Hankel singular values at σ_1 and σ_2 for $g_1 = -1.3265$ and $g_2 = 3.475$, i.e., the model is

$$F = \begin{bmatrix} 1.8265 & 1.3265 \\ -3.475 & 2.475 \end{bmatrix},\ G = \begin{bmatrix} -1.3265 \\ 3.475 \end{bmatrix},\ H = [\eta_0\ \eta_1].$$

The infinity norm of the approximation error is $0.101 = 2(\sigma_3 + \sigma_4)$, i.e., the upper

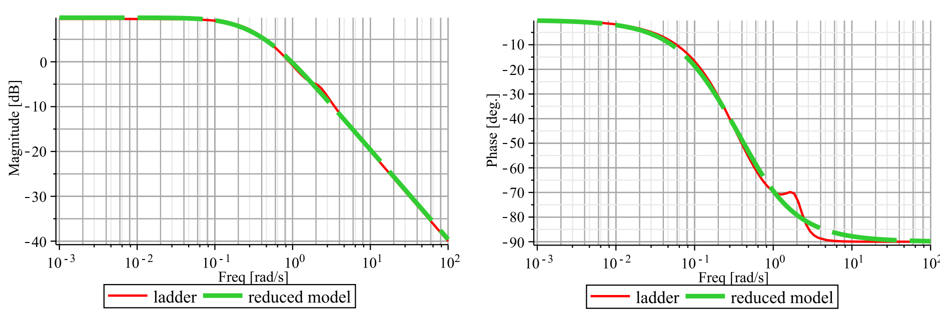

FIG. 3.2. *Bode plot of the ladder network v. a reduced model of order 2, with σ_1 and σ_2.*

bound of the balanced truncation infinity norm of the approximation error.

Note that we can define a class of models $(S - \bar{G}L, \bar{G}, \bar{H})$, which have the Hankel singular values σ_1 and σ_2, where $\bar{G}$ and $\bar{H}$ are parameterized in S and L. Furthermore we conjecture that all the models from this class approximate the given system with the same infinity norm of the error. □

Remark. Let $\nu = 1$ and consider the cross Gramian X of the asymptotically stable, minimal (scalar) system $(S - GL, G, H)$. The cross Gramian uniquely satisfies the Sylvester equation

$$(S - GL)X + X(S - GL) + GH = 0. \tag{3.1}$$

The unique solution is $X(G) = -\frac{H}{2}\frac{G}{S - GL}$.

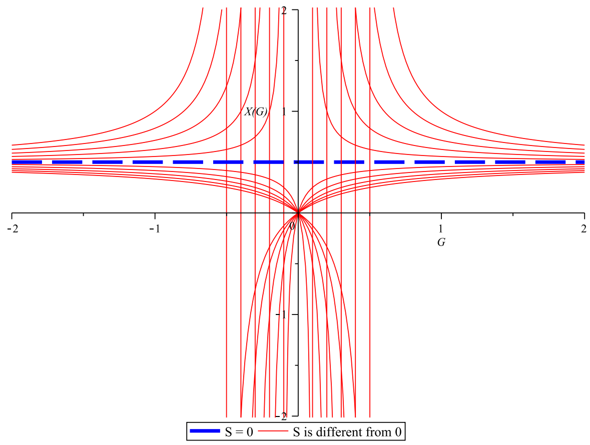

FIG. 3.3. *Plot of* $X(G)$*, for* $H>0$.

Note that for $S=0$, $X=-H/(2L)$, and does not depend on G, i.e., there is no G that assigns X to the Hankel singular value σ, see also Fig. 3.3. Let $\nu=2$ and consider a system of coordinates such that $(S-GL,G,H)$, with $S=\mathrm{diag}\{0,s_2\}$, $L=[1,\ 1]$, $H=[\eta_0\ \eta_1]$ and $G=[g_1\ g_2]^T\in\mathbf{G}$, $g_1\neq 0$, $g_2\neq 0$, has the cross Gramian $X=\mathrm{diag}\{\sigma_1,\sigma_2\}$. Let $v\in\mathbb{R}^2$ be the eigenvector of the 0 eigenvalue of S, i.e., $0=Sv$. Pre- and postmultiplying equation (3.1) with v yields $v^T(GLX+XGL-GH)v=0$, leading to $\det Q=0$, where $Q=GLX+XGL-GH$. We have

$$\begin{vmatrix} g_1(2\sigma_1-\eta_0) & g_1(\alpha-\eta_1)\\ g_2(\alpha-\eta_0) & g_2(2\sigma_2-\eta_1)\end{vmatrix}=0,$$

with $\alpha=\sigma_1+\sigma_2$, which further yields $\alpha^2+(\eta_0+\eta_1)\alpha+\sigma_1\eta_0+\sigma_2\eta_0-4\sigma_1\sigma_2=0$. It follows that $\mathrm{Tr}X=\sigma_1+\sigma_2=\alpha$ does not depend on G, i.e., there exist no G such that σ_1 and σ_2 can be modified. This line of thinking can be extended to $\nu>2$, where $\det Q=0$ becomes $F(\sigma_i)=0$, independent of G. □

4. Converse problem. Let $\nu=n=2$. Given a linear, minimal, asymptotically stable system (A,B,C) as in (2.1), find the (observable) pair (L,S), given by $L=[l_1\ l_2]$ and $S=\mathrm{diag}\{s_1,s_2\}$, and the parameter $G=[g_1\ g_2]^T$, , with $\nu=n=2$, such that the reduced order model $(S-GL,G,H)$ described by equations (2.5) is the balanced representation of (A,B,C), i.e., $W=M=\hat{W}=\hat{M}=\mathrm{diag}\{\sigma_1,\sigma_2\}$.

PROPOSITION 4.1. *Let system (2.1) be of order 2. Then system* $(S-GL,G,C\Pi)$*, with*

$$S=\mathrm{diag}\left\{\frac{1}{2}\frac{\eta_0^2(\sigma_1-\sigma_2)}{\sigma_1(\sigma_1+\sigma_2)},-\frac{1}{2}\frac{\eta_1^2(\sigma_1-\sigma_2)}{\sigma_2(\sigma_1+\sigma_2)}\right\},L=\begin{bmatrix}\frac{\eta_0}{\sigma_1+\sigma_2} & \frac{\eta_1}{\sigma_1+\sigma_2}\end{bmatrix},G=\begin{bmatrix}\eta_0\\ \eta_1\end{bmatrix}=H^T$$

satisfies the following properties: $(S-GL,G,H)$ *is balanced with the Hankel singular values* σ_1 *and* σ_2 *and matches the moments of* (A,B,C) *at* $\lambda(S)$*, up to a coordinate transformation.* □

Proof. The form of S, L and G is obtained by employing the balanced canonical form of the system (A,B,C) and solving an algebraic system of six equations in the unknowns $s_1,s_2,l_1,l_2,\eta_0,\eta_1$. The moments of the model $(S-GL,G,H)$ are $\eta_0/L_1=\eta_1/L_2=\sigma_1+\sigma_2$. According to [3], the model $(S-GL,G,H)$ matches the moments $C\Pi$ of the given system (A, B, C) if there exists P such that $HP = C\Pi$. Note that for $P = \mathrm{diag}\{L_1, L_2\}$ moment matching is satisfied, with $C\Pi = [\eta_0, \eta_1]$. ▯

Example (continued). The balanced reduced model of order two of the network ladder, with the Hankel singular values $\sigma_1 = 1.4895$ and $\sigma_2 = 0.0608$ is obtained for $S = \mathrm{diag}\{0.4763, -1.5103\}$, $L = [0.8004\ 0.2881]$, i.e.,

$$F = \begin{bmatrix} -0.5169 & -0.3576 \\ -0.3576 & -1.639 \end{bmatrix}, \quad H = [1.2409\ 0.4467] = G^T.$$

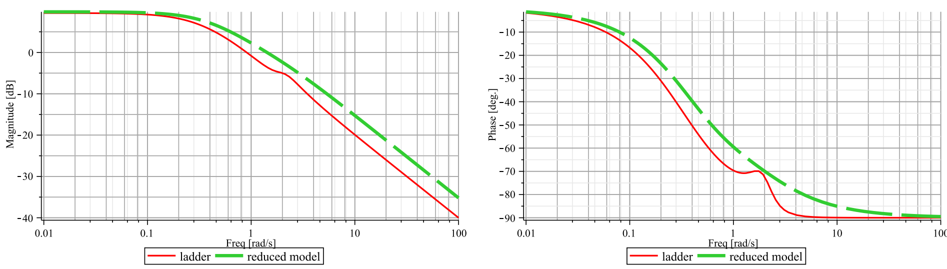

FIG. 4.1. *Bode plot of the ladder system v. the balanced model of order 2, with HSV's σ_1 and σ_2.*

The model is part of the class $(S - \bar{G}L, \bar{G}, \bar{H})$, defined in the first part of the example, i.e., it approximates the given system with an error with the infinity norm of 0.101. □

5. Hankel singular values as moments. In the previous sections we implicitly assumed that the Hankel singular values of (2.1) have already been computed. We turn the problem of computing the HSV's into a moment matching problem.

THEOREM 5.1. *Consider the system (A, B, C) and let $S \in \mathbb{R}^n$ and $L \in \mathbb{R}^{1\times n}$ be such that $\lambda(S) = \lambda(-A)$ and $L = B^T M$. Then the moments at $\lambda(S)$ contain information about the Hankel singular values of the system (A, B, C). If (A, B, C) is in balanced form, then the moments of (A, B, C) at $\lambda(S)$ are the squared Hankel singular values of the system, weighted by the entries of C, i.e., the moments are $c_1\sigma_1^2, \dots c_n\sigma_n^2$, where $C = [c_1 \dots c_n]$. Moreover, there exists a class of models $(S - GL, G, C\Pi)$ as in (2.5) that yield the same information about the HSV's of (A, B, C).* □

Proof. Multiplying equation (2.2) with M yields

$$AWM + WA^TM + BB^TM = 0. \tag{5.1}$$

Since $M > 0$, it means it is invertible, hence (5.1) becomes

$$A(WM) + B(B^TM) = -(WM)(-M^{-1}A^TM). \tag{5.2}$$

Denoting by $S = -M^{-1}A^TM$ and $L = B^TM$, the equation (5.2) becomes the Sylvester equation (2.4), with $\Pi = WM$. Since $\sigma(S) = \sigma(-A)$, we have that $\lambda(A) \cap \lambda(S) = \emptyset$. Furthermore, since the system (A, B, C) is assumed minimal, the pair (L, S)=$(B^TM, -M^{-1}A^TM)$ is observable. Hence, $\Pi = WM$ is the unique solution of the Sylvester equation (2.4) and furthermore, it is invertible, see e.g., [5]. The moments of the system (2.1) at $\lambda(-A)$ are given by $C\Pi = C(WM)$, see e.g., [3] and the first statement is proven.

Consider the balancing transformation $T \in \mathbb{R}^{n\times n}$. Applying the transformation T yields

$$TWT^T = T^{-T}MT^{-1} = \Sigma = \mathrm{diag}\{\sigma_1, ..., \sigma_n\}.$$

Note that $TWMT^{-1} = T\Pi T^{-1} = \Sigma^2$. Introducing these relations into equation (2.4), yields

$$\widetilde{A}\Sigma^2 + \widetilde{B}\widetilde{L} = \Sigma^2\widetilde{S}, \tag{5.3}$$

where $\widetilde{A} = T^{-1}AT$, $\widetilde{S} = T^{-1}ST$, $\widetilde{B} = T^{-1}B$ and $\widetilde{L} = LT$. Noting that $\lambda(\widetilde{S}) = \lambda(S) = \lambda(-A)$, the moments of (A, B, C) at $\lambda(S)$ are identical to the moments of $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ at $\lambda(\widetilde{S})$ and they are give by $\widetilde{C}\Sigma^2 = [c_1\sigma_1^2 \ ... \ c_n\sigma_n^2]$. Furthermore, a class of models that matches the moments of system (2.1) at $\sigma(A)$ are given by

$$\dot{\xi} = (\widetilde{S} - G\widetilde{L})\xi + Gu \tag{5.4}$$

$$\eta = C\Sigma^2\xi, \tag{5.5}$$

where $\xi(t) \in \mathbb{R}^n$ and G such that $\sigma(\widetilde{S} - G\widetilde{L}) \cap \sigma(\widetilde{S}) = \emptyset$. By the property of moment matching, the moments of these models contain direct information about the Hankel singular values of the system (2.1).□

Remark. Since the system (A, B, C) is assumed SISO, it is symmetric in the sense that the transfer function (scalar) is symmetric. Hence, if X is the cross Gramian of the system, then $X^2 = WM$ (see, e.g., [12]) and furthermore $|\lambda_i| = \sigma_i$, where $\lambda_i \in \lambda(X)$, $i = 1, ..., n$. Hence, the moments of (A, B, C) at $\lambda(-A)$ can be written as CX, contain information about the Hankel singular values of the system. □

6. Conclusions. In this paper we presented a moment matching problem associated with balancing a linear system and model reduction. We have described the subclass of reduced order models that achieve moment matching and have prescribed Hankel singular values. Then, by solving a converse moment matching problem for second order systems, we have obtained a second order model which achieves moment matching and is balanced with respect to a set of the first two Hankel singular values of the given system. Finally, we have related the Hankel singular values of the given system to the moments computed at the antistable modes of the given system.

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